GAUGE THEORY AND TOPOLOGY: AN INTRODUCTION
PRELIMINARY SYLLABUS

FRANCESCO LIN

Location and time. Università di Pisa, November-December 2025.

Contents. The goal of the course is to explore the interplay between topology and gauge theory, starting from the Dirac monopole (1931) up to the advent of Seiberg-Witten theory in low-dimensional topology (1994). The course will be divided in following the four parts (each taking roughly 6 hours):

(1) *The equations of gauge theory.* Magnetostatics in $\mathbb{R}^3$, the Aharonov-Bohm effect and the Dirac monopole. Connections on principal bundles and their curvature, Chern-Weil and Chern-Simons theory, the Yang-Mills(-Higgs) functional, Bogomolnyi bounds, ASD equations and the BPST instanton, BPS monopoles.
(2) *Analytic construction of moduli spaces.* Foundations of differential topology in infinite dimensions: Sobolev spaces, Fredholm operators, elliptic regularity, Hodge theory. Case study: the vortex equations and symmetric products of a surface.
(3) *Spin geometry.* Clifford algebras and modules, spin groups. Dirac operators and the Weitzenböck formula. The Atiyah-Singer index theorem for Dirac operators and its applications in geometry and topology.
(4) *Seiberg-Witten theory.* Seiberg-Witten equations (and why they are called monopole equations) and invariants. Sample applications to low-dimensional topology. Some words on Floer homology including discussion of why the symmetric product of a surface appears in Heegaard Floer homology.

I will post handwritten notes for the lectures; the plan is to eventually turn them into an introductory graduate level textbook for self-study, so any feedback is very welcome.

References. Some books we will follow are:

- Roe - Elliptic operators, asymptotic methods and topology.
- Donaldson, Kronheimer - The geometry of 4-manifolds.
- Morgan - The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds.
- Manton, Sutcliffe - Topological Solitons.
- Hamilton - Mathematical gauge theory.
- Jaffe, Taubes - Vortices and monopoles.
- Wald - Advanced classical electrodynamics.
- Shanahan - The Atiyah-Singer index theorem.

We will also refer to papers in the literature as the course progresses, for example:

- Garcia-Prada - A direct existence proof for the vortex equations over a compact Riemann surface.
- Kronheimer, Mrowka - The genus of embedded surfaces in the projective plane.

Prerequisites. First Master's level courses in analysis, differential geometry, differential and algebraic topology. Basic notions of classical electrodynamics will be helpful for context.

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