The Smale-Hirsch Theorem

An undergraduate thesis by
Emily Saunders
esaunders@college.harvard.edu
(207) 239-9156

Advised by Professor Michael J. Hopkins

Department of Mathematics
Harvard University
Cambridge, Massachusetts
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Contents

0 Fibrations

The following section is intended to serve as an introduction to the concept of fibrations. It will include basic definitions, as well as proofs of well-known facts about fibrations that will be used in the proof of the Smale-Hirsch theorem. This section can be skipped if the reader is already comfortable with fibrations.

Definition 0.1. *A map $p : E \to B$ has a the* ***homotopy lifting property*** *with respect to a space X if given a homotopy $h : X \times [0,1] \to B$ and an "initial lift" $\tilde{h}_0 : X \to E$ such that $p \circ \tilde{h}_0 = h(-,0)$, there exists a lift $\tilde{h} : X \times [0,1] \to E$ making the following diagram commute:*

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

Definition 0.2. *A map $p : E \to B$ is a* ***Hurewicz fibration*** *if it has the homotopy lifting property with respect to all spaces. p is a* ***Serre fibration*** *if it has the homotopy lifting property with respect to all CW-complexes.*

Note that to check a map has the homotopy lifting property with respect to all CW-complexes, it is sufficient to check that it has the property with respect to all disks, as one can then construct a lift on an arbitrary CW-complex by inductively lifting the k-skeleton.

Notice that these properties are concerned with maps into a base space B. It is then natural to consider how these maps interact and relate to one another. For instance, one might ask whether given $p : E \to B$ a Hurewicz fibration and $p' : E' \to B$ such that E is homeomorphic to E', is p' a Hurewicz fibration? The answer is not necessarily, because the key information lies in the map p', not the space E'. Thus we want to consider a notion of morphisms between these spaces that captures the importance of this data. Hence, when working with fibrations we will most commonly work in the following category:

Definition 0.3. ***The category of spaces over B*** *is the category consisting of objects given by continuous maps $p : E \to B$, and morphisms given by $f : E \to E'$ making the following commute,*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

which we will refer to as a ***map over B****.*

This leads us to our next important definitions which capture the version of homotopy and homotopy equivalence we want to use in this category,

Definition 0.4. *A* ***vertical homotopy*** *between two maps $f, g : E \to E'$ over B is a homotopy $H : E \times [0,1] \to E'$ between f, g that is a map over B.*

Definition 0.5. *A* ***fiberwise homotopy equivalence*** *between $p : E \to B$ and $p' : E' \to B$ is a map $f : E \to E'$ over B that admits a homotopy inverse $g : E' \to E$ over B such that the homotopies between $f \circ g$ and $Id_{E'}$, and $g \circ f$ and Id_E are vertical homotopies.*

Now we can return to our question above with a slight modification.

Proposition 0.1. *If $p : E \to B$ has the homotopy lifting property with respect to X and $p' : E' \to B$ is homeomorphic over B to $p : E \to B$, then p' has the homotopy lifting property with respect to X.*

Proof. Let $f : E \to E'$ be a homeomorphism over B. Then given a lifting problem for p' we can construct the following diagram,

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d] & E' \arrow[r, shift left, "f^{-1}"] \arrow[d, "p'"] & E \arrow[l, shift left, "f"] \arrow[dl, "p"] \\
X \times [0,1] \arrow[r, "h"] \arrow[urr, dashed, "\tilde{h}"] & B &
\end{tikzcd}$$

One can check that $f \circ \tilde{h}$ gives a lift of our original problem. ☐

We now are ready to move on to some theorems and definitions that will appear throughout the proof of the Smale-Hirsch theorem. First we have the following very important and surprising result that tells us this global property can in fact be checked locally.

Theorem 0.1. *If $p : E \to B$ is such that B admits a numerable open cover $\{U_\alpha\}_{\alpha \in A}$ where $p|_{p^{-1}(U_\alpha)} : p^{-1}(U_\alpha) \to U_\alpha$ has the homotopy lifting property with respect to a space X, then p has the homotopy lifting property with respect to X.*

To give a proof of this theorem without heavier machinery than I wish to assume in this thesis is very long and tedious, so I won't include it here. However, if interested, one can refer to Albrecht Dold's 1963 paper *Partitions of Unity in the Theory of Fibrations*, theorem 4.8 [2].

As a corollary to this theorem, we obtain one of the most important examples of Hurewicz fibrations,

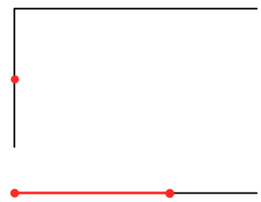
Corollary 0.1. *If $p : E \to B$ is a numerable fiber bundle, then p is a Hurewicz fibration.*

Proof. Choose a numerable open cover $\{U_\alpha\}_{\alpha \in A}$ such that for all α, $p^{-1}(U_\alpha)$ is homeomorphic over B to $\pi : F \times U_\alpha \to B$ for some space F, where π is the standard projection map. Now by proposition 0.2 it is sufficient to show that $F \times U_\alpha$ is a Hurewicz fibration. Given a lifting problem,

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d] & F \times U_\alpha \arrow[d, "\pi"] \\
X \times [0,1] \arrow[r, "h"] & U_\alpha
\end{tikzcd}$$

One can check that the map given by $(x,t) \mapsto (\pi_F(\tilde{h}_0(x,0)), h(x,t))$ gives a lift, where π_F is the projection $F \times U_\alpha \to F$. ☐

Now, we have seen above that the homotopy lifting property is homeomorphism-over-B invariant. We could then ask, is it fiberwise homotopy invariant? The answer to this question is no. For instance, the following image illustrates a space that is clearly fiberwise homotopy equivalent to $id : B \to B$, however the red path cannot be lifted with the given initial point.



This leads us to considering the following, weaker definition,

Definition 0.6. *A map $p : E \to B$ has the **weak homotopy lifting property** with respect to X if given a lifting problem*

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{h}_0} & E \\ \downarrow & & \downarrow p \\ X \times [0,1] & \xrightarrow{h} & B \end{array}$$

There exists a homotopy $\tilde{h} : X \times [0,1] \to E$ such that $p \circ \tilde{h} = h$ and a vertical homotopy $F : X \times [0,1] \to E$ between $\tilde{h}_0$ and $\tilde{h}(-,0)$

Proposition 0.2. *The weak homotopy lifting property is fiberwise homotopy invariant*

Proof. Let $p : E \to B$ be a weak Hurewicz fibration, $p' : E' \to B$ another space over B, and let $F : E \to E'$ be a fiberwise homotopy equivalence with homotopy inverse given by $G : E \to E'$. Suppose that we have a lifting problem,

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{h}_0} & E' \\ \downarrow & & \downarrow p' \\ X \times [0,1] & \xrightarrow{h} & B \end{array}$$

Post-composing with F gives us a new lifting problem for which we can find a lift:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d] & E' \arrow[r, "F"] \arrow[d, "p'"] & E \arrow[dl, "p"] \\ X \times [0,1] \arrow[r, "h"] \arrow[urr, "\tilde{h}"] & B & \end{tikzcd}$$

I claim that $G \circ \tilde{h}$ gives a weak lift of the original problem. First, we need to check that $p' \circ G \circ \tilde{h} = h$. Because G is a map over B, we know that $p' \circ G \circ \tilde{h} = p \circ \tilde{h} = h$.

Next we need a vertical homotopy between $G \circ \tilde{h}(-,0)$ and $\tilde{h}_0$. We already have a vertical homotopy $L : E' \times [0,1] \to E'$ between $G \circ F$ and $Id_{E'}$ as well as a vertical homotopy $K : X \times [0,1] \to E$ between $\tilde{h}(-,0)$ and $F \circ \tilde{h}_0$. Thus $((\tilde{h}_0, Id_{[0,1]}) \circ L) * (K \circ G)$ provides a vertical homotopy between $G \circ \tilde{h}(-,0)$ and $\tilde{h}_0$. $\square$

Definition 0.7. *A map $p : E \to B$ is **weakly fiber homotopy trivial** if given every $b \in B$ there exists a neighborhood $b \in U_b$ such that $p|_{p^{-1}(U_b)} : p^{-1}(U_b) \to U_b$ is fiberwise homotopy equivalent to $\pi : F \times U_b \to U_b$.*

The following proposition will be a crucial lemma in the proof of the Smale-Hirsch theorem,

Proposition 0.3. *If $p : E \to B$ is weakly fiber homotopy trivial, and B is paracompact, then p is a weak Hurewicz fibration.*

Proof. Using the weakly fiber homotopy trivial property of p and proposition 0.2, we can construct an open cover $\{U_b\}_{b\in B}$ of B such that the restriction of p to each U_b is a weak Hurewicz fibration. Now, because B is paracompact, we can choose a numerable refinement of this open cover. A modification of theorem 0.1 then tells us that p is a weak Hurewicz fibration. $\square$

Finally, we move on the the last lifting property we will need for this thesis,

Definition 0.8. *A map $p : E \to B$ has the **micro-homotopy lifting property** with respect to a space X if given a lifting problem,*

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"] & B \end{tikzcd}$$

*There exists a neighborhood $X \times \{0\} \subset U \subset X \times [0,1]$ and a lift $\tilde{h} : U \to E$ such that $p \circ \tilde{h} = h|_U$ and $\tilde{h}(-,0) = \tilde{h}_0$. We say p is a **micro-Hurewicz (Serre) fibration** if it has the micro-homotopy lifting property with respect to all spaces (all CW-Complexes).*

The motivation for this definition comes from the following proposition:

Proposition 0.4. *If $p : E \to B$ is a Hurewicz fibration, and $U \subset E$ is open, then $p|_U : U \to B$ is a Hurewicz micro-fibration*

Proof. Suppose we are given a homotopy lifting problem,

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d] & U \arrow[d, "p|_U"] \\ X \times [0,1] \arrow[r, "h"] & B \end{tikzcd}$$

post composition with the inclusion $i : U \to E$ gives us a new lifting problem, which we know to have a solution,

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d] & U \arrow[r, "i"] \arrow[d, "p|_U"] & E \arrow[dl, "p"] \\ X \times [0,1] \arrow[r, "h"] \arrow[urr, dashed, "\tilde{h}"] & B & \end{tikzcd}$$

Define $U' = \tilde{H}^{-1}(U)$. Then $\tilde{h}|_{U'}$ is a micro-lift. $\square$

Note that this proposition is also true of Serre fibrations.

Theorem 0.2. *If $p : E \to B$ is a Serre microfibration and a weak Serre fibration, then it is a Serre fibration*

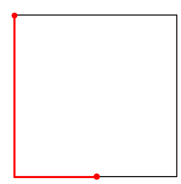
Proof. This proof is directly adapted from Michael Weiss's paper *Immersion Theory for Homotopy Theorists*[9].

We begin with a lifting problem,

$$\begin{tikzcd} \mathbb{D}^i \times \{0\} \arrow[r, "\tilde{w}_0"] \arrow[d] & E \arrow[d, "p"] \\ \mathbb{D}^i \times [0,1] \arrow[r, "w"] & B \end{tikzcd}$$

By the weak homotopy lifting property we obtain a weak lift $\tilde{w} : \mathbb{D}^i \times [0,1] \to E$ and a vertical homotopy $W : \mathbb{D}^i \times [0,1] \to E$ with $W(-,0) = \tilde{w}(-,0)$ and $W(-,1) = \tilde{w}_0$.

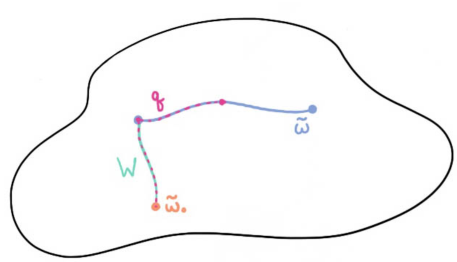
Define $L_{\frac{1}{2}} = \{(x,y) \in [0,1] \times [0,1] | xy = 0, x \leq \frac{1}{2}\}$ (see picture)



We are going to define a map $q : \mathbb{D}^i \times L_{\frac{1}{2}} \to E$ to connect our vertical homotopy with the first half of our lifted homotopy. Explicitly this map will be given by,

$$q(d,x,y) = \begin{cases} W(d,y) & x = 0 \\ \tilde{w}(d,x) & y = 0 \end{cases}$$

Note that $p \circ q(d,x,y) = w(d,x)$. The following image illustrates this map when $i = 0$,



We will now construct a new lifting problem by defining

$$Q : \mathbb{D}^i \times L_{\frac{1}{2}} \times [0, \frac{1}{2}] \to B$$

$$(d,x,y,t) \mapsto w(d,x+t)$$

And taking q to be our initial lift. We can use the micro homotopy lifting property to obtain a micro-lift $\tilde{Q} : U \to E$. By the compactness of $\mathbb{D}^i \times L_{\frac{1}{2}}$, there exists some $\delta \le \frac{1}{2}$ such that $\mathbb{D}^i \times L_{\frac{1}{2}} \times [0,\delta] \subset U$. For simplicity we will write $\tilde{Q} : \mathbb{D}^i \times L_{\frac{1}{2}} \times [0,\delta] \to E$ to be the restriction of our original $\tilde{Q}$

Now we will define another map $\Phi : \mathbb{D}^i \times [0,\delta] \times [0,1] \to \mathbb{D}^i \times L_{\frac{1}{2}} \times [0,\delta]$,

$$\Phi(d,s,t) = \begin{cases} (d,0,t-s,s) & t \ge s \\ (d,s-t,0,t) & s \ge t \end{cases}$$

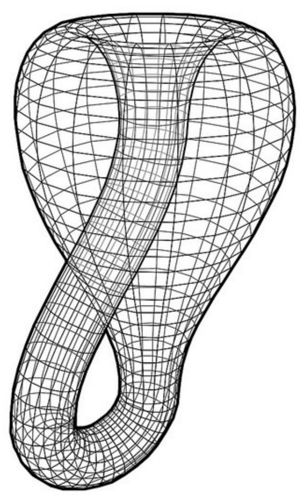
which we will use to combine our weak lift and our micro lift into a lift of the original problem. Define, $\tilde{w} : \mathbb{D}^i \times [0,1] \to E$ by

$$\tilde{w}(d,t) \mapsto \begin{cases} \tilde{Q} \circ \Phi(d,t,1-\frac{t}{\delta}) & t \in [0,\delta] \\ q(z,t,0) = \tilde{w}(d,t) & t \in [\delta,1] \end{cases}$$

One can check that this map is continuous and gives a lift of our original lifting problem. □

1 Introduction

Geometric intuition can be an incredibly powerful tool for a mathematician, yet it certainly has its limits. For instance, when one attempts to visualize a smooth manifold, we are attempting to imagine it smoothly embedding into the world we are able to picture, 3-dimensional or lower Euclidean space, however the sampling of manifolds that we are actually able to smoothly embed into 3-space is relatively minuscule. Luckily, we have many tools that allow us to push beyond this barrier and attempt to visually understand higher-dimensional, more complicated objects. One of the most important such tools is considering how manifolds can be immersed. Many properties of manifolds which we consider to be important are local. Thus an immersion, which is a smooth map that is locally a smooth embedding, allows us to visualize a great deal of important information we may not otherwise be able to picture. Through this we can think far more intuitively about spaces that we will never truly be able to see embedded into dimensions we can visualize such as the projective plane and the Klein bottle. For instance, when looking at the immersion of the Klein bottle illustrated below, one can imagine being an ant on the surface and traveling along the bottle. We can see paths that would take us from one side of the surface to the other, which allows us to visualize what it means for this surface to be non-orientable.



This thesis aims to present a proof of one of the most important results in immersion theory, the Smale-Hirsch theorem. This theorem helps inch us closer to the questions of when can we immerse one manifold into another, and how many different ways can we do so?

In 1959 Stephen Smale made a significant contribution to these questions by classifying the immersions of the 2 sphere into $\mathbb{R}^n$, $n \geq 3$[6]. In particular, this classification gave us the somewhat surprising result that there is only one immersion of the sphere into $\mathbb{R}^3$ up to regular homotopy, thus the sphere can be turned inside-out via a regular homotopy. Smale then generalized this classification to classifying the immersions of S^n into $\mathbb{R}^q$ for $n \geq 2, q \geq n+1$ [7]. The tools Smale developed in these proofs were later generalized further to many powerful geometric results [8]. In particular, Mikhail Gromov and Yakov Eliashberg developed tools to solve a number of different geometric problems concerning open differential relationships in jet spaces (of which the Smale-Hirsch theorem is an example) [3].

The Smale-Hirsch theorem plays an crucial role in answering these questions because it allows us to reduce computations about spaces of immersions to ones concerning continuous maps and vector bundles. For instance, in 1985 Ralph L. Cohen published a proof of the immersion conjecture [1], which states,

Theorem 1.1 (Immersion Conjecture)**.** *Let M be a smooth manifold of dimension n, and let $\alpha(n)$ be the number of 1s in the base 2 expression of n. Then there exists an immersion of M into $\mathbb{R}^{2n-\alpha(n)}$.*

This conjecture provides a significant improvement to Whitney's 1943 bound, which states that a smooth n-manifold can be immersed into $\mathbb{R}^{2n-1}$[10]. The Smale-Hirsch theorem plays an important role in Cohen's proof of the immersion conjecture, as it allows us to reduce to computations about vector bundle monomorphisms on our manifolds, which we are far better equipped to understand with homotopy-theoretic tools.

2 The Smale-Hirsch Theorem

The statement of the Smale-Hirsch theorem is as follows:

Theorem 2.1 (Hirsch-Smale)**.** *Let M^m, N^n be smooth manifolds where $m < n$. Suppose M is compact, N without boundary. Let $\boldsymbol{imm}(M,N)$ be the space of smooth immersions $M \to N$, and $\boldsymbol{fimm}(M,N)$ be the space of formal immersions (i.e. pairs $(f, \delta f)$ where $f : M \to N$ is continuous, and δf is a monomorphism of tangent bundles $TM \to f^*TN$). Then the map,*

$$\boldsymbol{imm}(M,N) \to \boldsymbol{fimm}(M,N)$$

$$f \mapsto (f, df)$$

Is a weak homotopy equivalence.

While spaces of immersions can be difficult to understand, this theorem allows us to reduce to studying spaces which we have far better tools to compute. For instance, if we are looking to compute the space of immersions of a compact manifold M into $\mathbb{R}^n$ up to regular homotopy, or in other words, $\pi_0(\mathbf{imm}(M, \mathbb{R}^n))$, the Smale-Hirsch theorem tells us that it is sufficient to compute $\pi_0(\mathbf{fimm}(M, \mathbb{R}^n))$. The space $\mathbf{fimm}(M, \mathbb{R}^n)$ is a fiber bundle over $\mathbf{Map}(M, \mathbb{R}^n)$ under the canonical projection $(f, \delta f) \mapsto f$. The space $\mathbf{Map}(M, \mathbb{R}^n)$ is contractible, which tells us that $\mathbf{fimm}(M, \mathbb{R}^n)$ is homotopy equivalent to the fiber over a point. Choosing this point to be the identity map, we have now reduced this problem to computing π_0 of the space of vector bundle monomorphisms $TM \to M \times \mathbb{R}^n$. If such a monomorphism were to exist, we could choose a metric and take the orthogonal compliment of the image. Thus as a corollary to the Smale-Hirsch theorem we have the following criteria for the existence of an immersion $M \to \mathbb{R}^n$ [5]:

Corollary 2.1. *For a compact manifold M of dimension $m < n$, there exists an immersion $M \to \mathbb{R}^n$ if and only if there exists a $n-m$ dimensional vector bundle over M such that $M \times \mathbb{R}^n \cong V \oplus TM$*

The primary source for this proof is the paper *Immersion Theory for Homotopy Theorists* by Michael Weiss. Weiss's proof employs a very interesting and surprising application of category theory in the proof of the most challenging lemma (section 5 in this thesis), which beautifully translates this geometric problem to the language of category theory. This use of category theory allows us to organize a messy geometric problem into constructing a category-like structure on the fibers of a map which provides exactly the data we need to prove this key lemma.

2.1 Reduction of the Theorem

Our first step will be to reduce the proof of the Smale-Hirsch theorem to the following:

Theorem 2.2. *Let M and N be smooth manifolds, both of dimension n, where M is compact with no closed components, and N has empty boundary. Then the map*

$$\boldsymbol{imm}(M,N) \to \boldsymbol{fimm}(M,N)$$

is a weak homotopy equivalence

This statement may at first appear to be weaker than the statement of the Smale-Hisch theorem, however the intuition lies in the fact that given an immersion of a manifold $M \to N$ where $dim(M) < dim(N)$, and M is compact, we can extend this to an immersion of a disk bundle on M of dimension $dim(N) - dim(M)$.

Lemma 2.1. *Theorem 2.2 implies theorem 2.1.*

Proof. First we will prove the following; for a fixed vector bundle $V \to M$ of dimension $m - n$, and $\mathbb{D}(V)$ the corresponding disk bundle, the following diagram is a weak homotopy pullback square.

$$\begin{array}{ccc} \mathbf{imm}(\mathbb{D}(V), N) & \longrightarrow & \mathbf{fimm}(\mathbb{D}(V), N) \\ \downarrow & & \downarrow \\ \mathbf{imm}(M, N) & \longrightarrow & \mathbf{fimm}(M, N) \end{array}$$

With vertical arrows the restriction maps to the 0 section of $\mathbb{D}(V)$.

Define the space $\mathbf{imm}_V(M, N)$ to be the space of pairs (g, ι) where $g : M \to N$ is an immersion and ι is an isomorphism of V with the normal bundle of g. Similarly, we define the space $\mathbf{fimm}_V(M, N)$ to be the space of triples $(f, \delta f, \iota)$ where $(f, \delta f) \in \mathbf{fimm}(M, N)$ and ι is a vector bundle isomorphism $V \to f^*TN/Im(\delta f)$.

We need the following two lemmas,

Lemma 2.2. *The map*

$$\boldsymbol{imm}(\mathbb{D}(V), N) \to \boldsymbol{imm}_V(M, N)$$

$$f \mapsto (f|_M, \iota)$$

where ι denotes the isomorphism of V with the normal bundle of f induced by df, is a homotopy equivalence.

Lemma 2.3. *The map*

$$\boldsymbol{fimm}(\mathbb{D}(V), N) \to \boldsymbol{fimm}_V(M, N)$$

$$(f, \delta f) \mapsto (f|_M, \delta f|_{TM}, \iota)$$

where ι is the isomorphism induced by δf, is a weak homotopy equivalence.

To see a sketch of the proof of these lemmas see lemma 5.3 in Michael Weiss's paper [9].

By Lemmas 2.2 and 2.3 we can replace the top row to get,

$$\begin{array}{ccc} \mathbf{imm}_V(M, N) & \longrightarrow & \mathbf{fimm}_V(M, N) \\ \downarrow & & \downarrow \\ \mathbf{imm}(M, N) & \longrightarrow & \mathbf{fimm}(M, N) \end{array}$$

This is clearly a pullback square. Therefore all that remains to be shown is that $\mathbf{imm}_V(M, N) \to \mathbf{imm}(M, N)$ is a Serre fibration. This follows from the fact that it is a fiber bundle (corollary 0.1).

Now, with M as in theorem 2.1, select a point $(f, \delta f) \in \mathbf{fimm}(M, N)$. Set $V = Coker(\delta f)$. This tells us that in the above diagram, some element of $\mathbf{fimm}(\mathbb{D}(V), N)$ maps into the connected component containing $(f, \delta f)$. If we assume the statement of theorem 2.2, the top arrow of the above diagram is a weak homotopy equivalence. Thus because the diagram is a weak homotopy pullback square, the bottom arrow is as well on the connected component containing $(f, \delta f)$. Note however, that our choice of $(f, \delta f)$ was arbitrary. Therefore the bottom arrow is a weak homotopy equivalence. □

The following corollary will now provide us with our base case for the inductive argument that will complete the proof:

Corollary 2.2. *For N a smooth n-dimensional manifold without boundary, the map,*

$$\boldsymbol{imm}(\mathbb{D}^n, N) \to \boldsymbol{fimm}(\mathbb{D}^n, N)$$

is a weak homotopy equivalence.

Proof. If we take $M = pt$ in the above diagram, we have that the following is a weak homotopy pullback square,

$$\begin{tikzcd} \mathbf{imm}(\mathbb{D}^n, N) \arrow[r] \arrow[d] & \mathbf{fimm}(\mathbb{D}^n, N) \arrow[d] \\ N \arrow[r, "Id"] & N \end{tikzcd}$$

Thus because the bottom arrow is clearly a weak homotopy equivalence (as it is a homeomorphism), the top arrow is as well. ☐

We will now spend the remainder of this paper proving theorem 2.2.

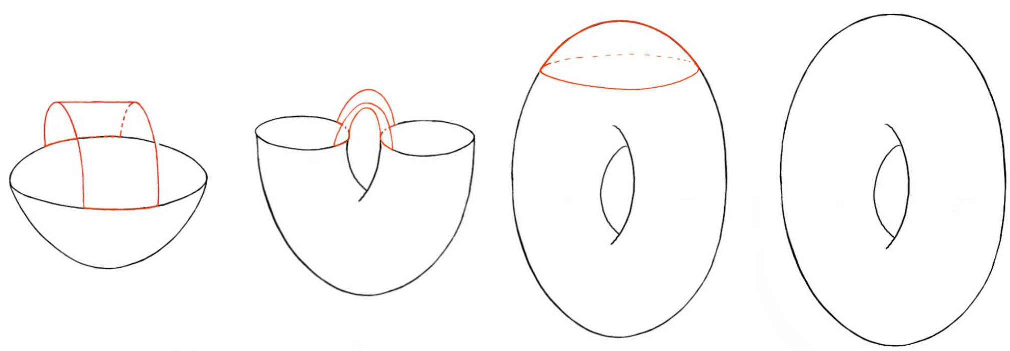
2.2 Handle Decompositions

Definition 2.1. *A handle decomposition of a manifold M is a filtration,*

$$\emptyset = L_0 \subset L_1 \subset ... \subset L_n = M$$

Where L_i is obtained by attaching to L_{i-1} a copy of $\mathbb{D}^k \times \mathbb{D}^{n-k}$ (a handle of index k) via an embedding of $S^{k-1} \times \mathbb{D}^{n-k}$ to ∂L_{i-1}.

The following image is a handle decomposition of the torus with 1 handle of index 0, 2 handles of index 1 and 1 handle of index 2:



The final step of the proof of this theorem will be given by an inductive argument on a handle decomposition of our manifold M. However this inductive argument will require a lemma that is only true for handles of index $< n$. Therefore we must prove the following lemma,

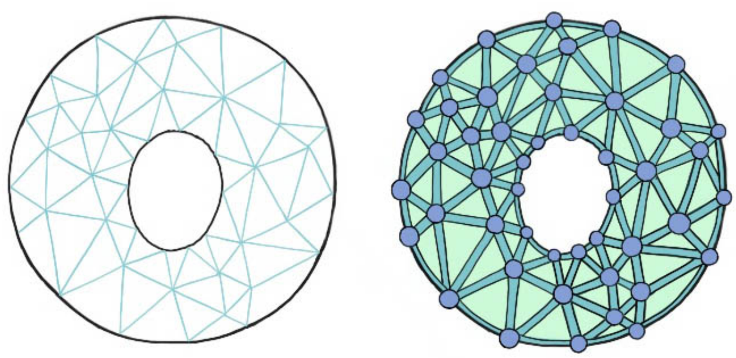
Lemma 2.4. *Let M be a smooth compact manifold of dimension n with no closed components. Then L admits a handle decomposition with no handles of index n.*

Note that it is crucial that M have no closed components, as this lemma fails in the case where M is the 2-sphere (it is a fun exercise to attempt to draw such a decomposition and see what goes wrong).

Proof. Choose a smooth triangulation $X \to M$. It is a well-known fact that each smooth manifold admits such a triangulation and we will assume it here. Now choose a filtration of X such that X_i contains exactly one more simplex than X_{i-1}. This is finite by the compactness of M. Let L_i be a regular neighborhood of X_i. Then the filtration,

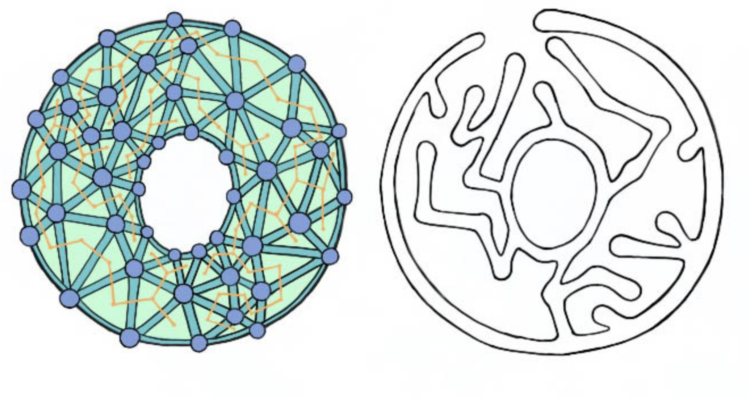
$$\emptyset = L_0 \subset L_1 \subset ... \subset L_k \subset M$$

Is a handle decomposition of M, with a handle of index i for each i simplex. The following image illustrates and example of a smooth triangulation on a manifold and the resulting handle decomposition,



Now to finish the proof we want to remove the handles in this decomposition of index n. To do this we construct a graph Γ with vertices the barycenters of the n simplices of X and edges between the barycenters of adjacent n simplices. This is a connected graph, and we choose a spanning tree T. Here is where we require that M not have empty boundary. Choose some vertex b that is the barycenter of a simplex meeting ∂M. For each additional vertex $q \neq p$ there is a unique minimal path from q to p. This path intersects a unique face of the n-simplex corresponding to q, denote this $e(q)$, and denote the number of edges in this path as $d(q)$.

We now remove handles in pairs. First we remove the n-simplex corresponding to p along with a face that is a part of the boundary. We then remove the n-simplices with $d(q) = 1$ along with $e(q)$. We inductively continue this process until there are no n-simplicies remaining. Note that at each step, the diffeomorphism type of the resulting manifold is unchanged. A graph corresponding to the above example and resulting manifold are illustrated below.



□

2.3 Main Lemma

This section will be devoted to proving the following lemma which will allow us to preform an inductive argument on the handle decomposition of our manifold in the final section of this proof:

Lemma 2.5. *Let $A^k = \{v \in \mathbb{R}^k | \frac{1}{2} \leq ||v|| \leq 1\}$. Then the restriction map,*

$$r : \textbf{\textit{(f)imm}}(\mathbb{D}^k \times \mathbb{D}^{n-k}, N) \to \textbf{\textit{(f)imm}}(A^k \times \mathbb{D}^{n-k}, N)$$

is a Serre fibration when $k \geq 1$.

Proof. First we will begin with proving this lemma for the space of formal immersions, as this is the easier step.

Suppose that we are given a lifting problem,

$$\begin{tikzcd} \mathbb{D}^i \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d] & \textbf{fimm}(D^k \times \mathbb{D}^{n-k}, N) \arrow[d, "r"] \\ \mathbb{D}^i \times [0,1] \arrow[r, "H"] & \textbf{fimm}(A^k \times \mathbb{D}^{n-k}, N) \end{tikzcd}$$

To construct a lift we are going to break down the data of this diagram. Note that a map $A \to Map(B, C)$ is equivalent to the data of a map $A \times B \to C$. Thus, the data of our homotopy H is equivalent to a map $H' : \mathbb{D}^i \times [0,1] \times A^k \times \mathbb{D}^{n-k} \to N$ along with an isomorphism of vector bundles $\varphi : \pi^*(TA^k \times \mathbb{D}^{n-k}) \to H'^*(TN)$ where π is the projection $\mathbb{D}^i \times [0,1] \times A^k \times \mathbb{D}^{n-k}$. The data of $\tilde{h}_0$ gives us an extension of our map H' to $\mathbb{D}^i \times [0,1] \times A^k \times \mathbb{D}^{n-k} \cup \mathbb{D}^i \times \{0\} \times \mathbb{D}^k \times \mathbb{D}^{n-k} \subset \mathbb{D}^i \times [0,1] \times \mathbb{D}^k \times \mathbb{D}^{n-k}$. The first part of constructing a lift of our original problem is extending this map to all of $\mathbb{D}^i \times [0,1] \times \mathbb{D}^k \times \mathbb{D}^{n-k}$. In other words, we want to find a map making the following diagram commute:

$$\begin{tikzcd}
 & \mathbb{D}^i \times \{0\} \times \mathbb{D}^k \times \mathbb{D}^{n-k} \arrow[d] \arrow[dr, "\tilde{h}_0'"] & \\
\mathbb{D}^i \times \{0\} \times A^k \times \mathbb{D}^{n-k} \arrow[ur] \arrow[r] \arrow[dr] & \mathbb{D}^i \times [0,1] \times \mathbb{D}^k \times \mathbb{D}^{n-k} \arrow[r, dashed] & N \\
 & \mathbb{D}^i \times [0,1] \times A^k \times \mathbb{D}^{n-k} \arrow[u] \arrow[ur, "H'"'] &
\end{tikzcd}$$

Notice that the space $\mathbb{D}^i \times [0,1] \times A^k \times \mathbb{D}^{n-k} \cup \mathbb{D}^i \times \{0\} \times \mathbb{D}^k \times \mathbb{D}^{n-k}$ is a retract of $\mathbb{D}^i \times [0,1] \times \mathbb{D}^k \times \mathbb{D}^{n-k}$. Letting p be the retraction, $(H' \cup \tilde{h}_0) \circ p$ gives us an extension of our map.

$\tilde{h}_0$ combined with H defines the data of a vector bundle isomorphism $\tilde{\varphi} : \pi'''^*(T\mathbb{D}^k \times \mathbb{D}^{n-k}) \to (H' \cup \tilde{h}_0')^*(TN)$, where π''' is the projection of $\mathbb{D}^i \times [0,1] \times A^k \times \mathbb{D}^{n-k} \cup \mathbb{D}^i \times \{0\} \times \mathbb{D}^k \times \mathbb{D}^{n-k}$, that is compatible with φ. We want to extend this to a vector bundle isomorphism $\tilde{\varphi} : \pi''^*(T\mathbb{D}^k \times \mathbb{D}^{n-k}) \to ((H' \cup \tilde{h}_0) \circ p)^*(TN)$ where π'' is the projection $\mathbb{D}^i \times [0,1] \times \mathbb{D}^k \times \mathbb{D}^{n-k} \to \mathbb{D}^k \times \mathbb{D}^{n-k}$.

First note that the following diagram commutes,

$$\begin{tikzcd}
\mathbb{D}^i \times [0,1] \times A^k \times \mathbb{D}^{n-k} \arrow[r] \arrow[dd, "\pi"] & \mathbb{D}^i \times [0,1] \times \mathbb{D}^k \times \mathbb{D}^{n-k} \arrow[d, "p"] \arrow[ddr, bend left, "\pi''"] & \mathbb{D}^i \times \{0\} \times \mathbb{D}^k \times \mathbb{D}^{n-k} \arrow[l] \arrow[dd, "\pi'"] \\
 & \mathbb{D}^i \times [0,1] \times A^k \times \mathbb{D}^{n-k} \cup \mathbb{D}^i \times \{0\} \times \mathbb{D}^k \times \mathbb{D}^{n-k} \arrow[dr, "\pi'''"] & \\
A^k \times \mathbb{D}^{n-k} \arrow[rr] & & \mathbb{D}^k \times \mathbb{D}^{n-k}
\end{tikzcd}$$

Our isomorphism $\tilde{\varphi}$ induces an isomorphism $\tilde{\varphi}' : (\pi''' \circ p)^*(T\mathbb{D}^k \times \mathbb{D}^{n-k}) \to ((H' \cup \tilde{h}_0) \circ p)^*(TN)$. The commutivity of the above diagram tells us that $(\pi''' \circ p)^*(T\mathbb{D}^k \times \mathbb{D}^{n-k}) \cong \pi''^*(T\mathbb{D}^k \times \mathbb{D}^{n-k})$ so this gives us the isomorphism we need.

To prove the lemma for the space of immersions, we will begin by noting the following facts about Serre fibrations from section 0:

1. If $p : E \to B$ is a Serre fibration, and $U \subset E$ is open, then $p|_U : U \to B$ is a Serre microfibration (Proposition 0.4).

2. If $p : E \to B$ is a Serre microfibration and a weak Serre fibration, then it is a Serre fibration (Theorem 0.2).

3. If $p : E \to B$ is weakly fiber homotopy trivial (meaning for every $b \in B$ there exists a neighborhood $b \in U_b$ and vertical homotopy equivalence $p^{-1}(U_b) \to F \times U_b$) and B is paracompact, then it is a weak fibration (Proposition 0.3).

Thus the proof of this lemma will be broken into 2 parts, first demonstrating that this restriction map is a Serre microfibration, then applying property 3 and a category-like structure to demonstrate that it is a Serre fibration.

Lemma 2.6. *Let $M_0 \subset M$ be a compact codimension 0 submanifold with the following boundary condition: M_0 locally looks like $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n-2} \subset \mathbb{R}_+ \times \mathbb{R}^{n-1}$ i.e. there are corners if and only if M_0 intersects ∂M. The restriction map $\rho : C^\infty(M,N) \to C^\infty(M_0,N)$ is a Serre fibration.*

The proof of this lemma is straightforward but long, so I will not include it here. For a detailed exposition see lemma 4.4 Michael Weiss's paper [9].

Corollary 2.3. *The restriction map*

$$r : \boldsymbol{imm}(\mathbb{D}^k \times \mathbb{D}^{n-k}, N) \to \boldsymbol{imm}(A^k \times \mathbb{D}^{n-k}, N)$$

is a Serre microfibration.

By property 1, this follows immediately from the lemma and the fact that for a compact M, $\mathbf{imm}(M, N)$ is an open subset of $C^\infty(M, N)$.

Now, we are attempting to prove that this map is a Serre fibration, thus we are attempting to lift a map of the form,

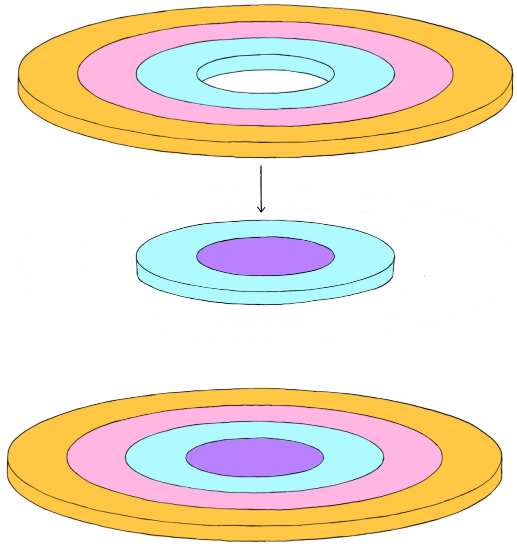
$$\begin{array}{ccc} \mathbb{D}^i \times \{0\} & \xrightarrow{\tilde{f}_0} & \mathbf{imm}(\mathbb{D}^k \times \mathbb{D}^{n-k}) \\ \downarrow & & \downarrow \\ \mathbb{D}^i \times [0,1] & \xrightarrow{f} & \mathbf{imm}(A^k \times \mathbb{D}^{n-k}) \end{array}$$

Note that we can form a pullback square and obtain a lifting problem,

$$\begin{array}{ccccc} \mathbb{D}^i \times \{0\} & \xrightarrow{i \times \tilde{f}_0} & (\mathbb{D}^i \times [0,1]) \underset{\mathbf{imm}(A^k \times \mathbb{D}^{n-k}, N)}{\times} \mathbf{imm}(\mathbb{D}^k \times \mathbb{D}^{n-k}, N) & \longrightarrow & \mathbf{imm}(\mathbb{D}^k \times \mathbb{D}^{n-k}, N) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}^i \times [0,1] & \xrightarrow{Id} & \mathbb{D}^i \times [0,1] & \xrightarrow{f} & \mathbf{imm}(A^k \times \mathbb{D}^{n-k}, N) \end{array}$$

If we are given a lift in the left hand square, then we obtain a lift of our initial problem, thus we can assume that our base space is a disk. Note that properties such as the micro homotopy lifting property are invariant under pullback squares, so we may still assume that our map is a microfibration. Furthermore, $\mathbb{D}^i \times [0,1]$ is paracompact, thus by property 3 it is sufficient to show that this map is weakly fiber homotopy trivial. For simplicity, we will use the notation $p : E \to B$ for this map.

At this point we are going to employ a category-like composition structure on the fibers of this map. Note that the fiber over a point is still a space of immersions $\mathbb{D}^k \times \mathbb{D}^{n-k} \to N$ that are fixed on $A^k \times \mathbb{D}^{n-k} \subset \mathbb{D}^k \times \mathbb{D}^{n-k}$. Notice that if we take an immersion the larger annulus $S^{k-1} \times [0,3] \times \mathbb{D}^{n-k}$ such that when we restrict to the inner ring $S^{k-1} \times [0,1] \times \mathbb{D}^{n-k}$ we obtain (up to reparameterization) an equivalent immersion to one given by restricting $\mathbb{D}^k \times \mathbb{D}^{n-k} \to N$ to $A^k \times \mathbb{D}^{n-k}$, we can glue this larger annulus to the disk and obtain a new immersion of the disk which, by adjusting the parameters, lives in the fiber over the immersion obtained by restricting to $S^{k-1} \times [2,3] \times \mathbb{D}^{n-k}$. This idea illustrated in the following image.



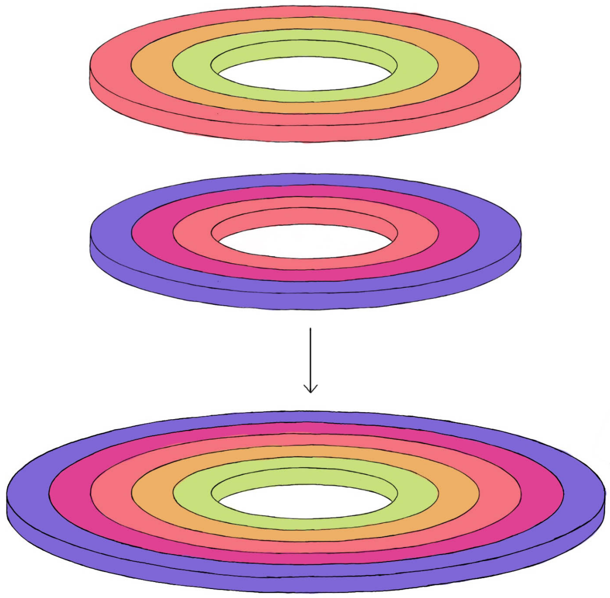
This structure is "category-like" because we can imagine the fibers over the points as objects and these immersions of the larger annulus as morphisms between them. We will see below that these will satisfy a composition law that will not be strictly associative or unital, however it will be so up to homotopy. For an immersions $f \in \mathbf{imm}(S^{k-1} \times [0,3] \times \mathbb{D}^{n-k}, N)$ and $g \in \mathbf{imm}(\mathbb{D}^k \times \mathbb{D}^{n-k}, N)$ we will use the notation $f * g$ to denote this morphism structure.

To prove that the space is weakly fiber homotopy trivial, for each point $b \in B$ we must find a neighborhood U_b and a homotopy equivalence $p^{-1}(U_b) \to F \times U_b$. This is equivalent to the data of a homotopy equivalence of fibers $p^{-1}(b) \to p^{-1}(b')$ for all $b' \in U_b$ that varies continuously in choice of b'. Thus we will use these immersions of the larger annulus to construct maps between the fibers that are continuous in our choice of points.

For ease of notation, we will call our space $\mathbf{imm}(S^{k-1} \times [0,3] \times D^{n-k}, N) = Z$. First we will construct the following 2 maps

$$\sigma, \tau : Z \to \mathbf{imm}(A^k \times \mathbb{D}^{n-k}, N)$$

such that σ is the restriction to $S^{k-1} \times [0,1] \times \mathbb{D}^{n-k}$ and τ is the restriction to $S^{k-1} \times [2,3] \times \mathbb{D}^{n-k}$. These functions are meant to represent selecting the domain and target of a given "morphism" of Z. Note that for points $z, z' \in Z$ such that $\sigma(z) = \tau(z')$ we can "compose" them via the following illustration to obtain a new morphism $z \circ z'$ that has $\sigma(z \circ z') = \sigma(z')$ and $\tau(z \circ z') = \tau(z)$.



First we will construct the "identity morphisms":

Lemma 2.7. *Let Δ be the diagonal of the space $\boldsymbol{imm}(A^k \times \mathbb{D}^{n-k}, N) \times \boldsymbol{imm}(A^k \times \mathbb{D}^{n-k}, N)$. There exists a map,*

$$\iota : \Delta \to Z$$

Such that $(\sigma, \tau) \circ \iota = Id_\Delta$ and there exists a vertical homotopy over the restriction map $r : \boldsymbol{imm}(\mathbb{D}^k \times \mathbb{D}^{n-k}, N) \to \boldsymbol{imm}(A^k \times \mathbb{D}^{n-k}, N)$

$$h : \boldsymbol{imm}(\mathbb{D}^k \times \mathbb{D}^{n-k}, N) \times [0,1] \to \boldsymbol{imm}(\mathbb{D}^k \times \mathbb{D}^{n-k}, N)$$

such that $h(f, 0) = \iota(r(f)) * f$ and $h(f, 1) = f$.

In words, this lemma states that there is a continuous function ι which picks out morphisms $r^{-1}(f) \to r^{-1}(f)$ that act as the identity up to vertical homotopy.

Proof. We begin by defining the thickened space $(A^k \times \mathbb{D}^{n-k})_+ = (\mathbb{R}^k - \{0\}) \times \mathbb{R}^{n-k}$. Let r' : $\mathbf{imm}((A^k \times \mathbb{D}^{n-k})_+, N) \to \mathbf{imm}(A^k \times \mathbb{D}^{n-k}, N)$ be the restriction map. We first want to prove that this map admits a section s.

To find such a section we first find a section of the restriction map,

$$C^\infty((A^k \times \mathbb{D}^{n-k})_+, N) \to C^\infty(A^k \times \mathbb{D}^{n-k}, N)$$

This exists by the fact that smooth functions are defined to be such that they extend to a smooth function on a neighborhood of each point, thus because $A^k \times \mathbb{D}^{n-k}$ is compact, we can find a tubular neighborhood on which we can extend a smooth function and contract our space $(A^k \times \mathbb{D}^{n-k})_+$ so that it maps into this neighborhood. It remains to show that this can be done continuously in your choice of smooth function. For a more detailed explanation of this see Michael Weiss's paper. Now we can restrict this section to a section,

$$s' : \mathbf{imm}(A^k \times \mathbb{D}^{n-k}, N) \to C^\infty((A^k \times \mathbb{D}^{n-k})_+, N))$$

Now note that we can find a continuous function,

$$\nu : (A^k \times \mathbb{D}^{n-k})_+ \times [0, \infty) \to (A^k \times \mathbb{D}^{n-k})_+$$

Such that ν_0 is the identity, and at time t, ν_t is a smooth embedding $(A^k \times \mathbb{D}^{n-k})_+ \to (A^k \times \mathbb{D}^{n-k})_+$ that is within distance ε_t from $A^k \times \mathbb{D}^{n-k}$ where $\varepsilon_t \to 0$ as $t \to \infty$. Note that the set,

$$U = \{(f, z) \in \mathbf{imm}(A^k \times \mathbb{D}^{n-k}, N) \times (A^k \times \mathbb{D}^{n-k})_+ | ds'(f)|_z \text{ is injective}\}$$

is an open subset. We construct a map,

$$\tau : \mathbf{imm}(A^k \times \mathbb{D}^{n-k}, N) \to [0, \infty)$$

Such that for a given f, the subset,

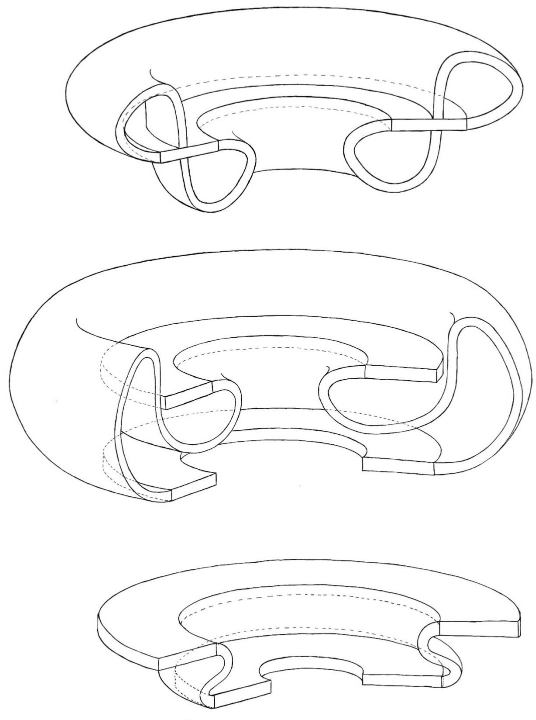
$$\{z \in (A^k \times \mathbb{D}^{n-k})_+ | (f, z) \in U\}$$

is within distance $\varepsilon_{\tau(f)}$ from $A^k \times \mathbb{D}^{n-k}$. Now we can construct our section to be $f \mapsto s'(f) \circ \nu_{\tau(f)}$. Call this section s.

At this point, it is sufficient to construct an immersion,

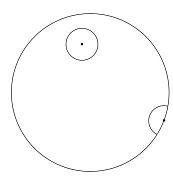
$$f : S^{k-1} \times [0, 3] \times \mathbb{D}^{n-k} \to (A^k \times \mathbb{D}^{n-k})_+$$

Such that $f|_{S^{k-1} \times [0,1] \times D^{n-k}} = f|_{S^{k-1} \times [2,3] \times \mathbb{D}^{n-k}}$ and homotopy h' between f and an immersion that maps $S^{k-1} \times [0, 3] \times \mathbb{D}^{n-k}$ smoothly onto $A^k \times \mathbb{D}^{n-k}$ and is f when restricted to $S^{k-1} \times [2, 3] \times \mathbb{D}^{n-k}$ for all times t, as we can then define ι to be $s \circ f$, and $h = s \circ h'$. I claim that it is sufficient to do this in the case where $k = 2$ and $n = 3$, as we can extend f with the identity in the remaining coordinates for higher values of n, and higher values of k can be obtained from $k = 2$ by taking cross sections. The case of $k = 2, n = 3$ is illustrated in the following image,



□

Now we continue with the proof of lemma 4.1. Fix $b \in B$. Because B is a disk, we can choose a neighborhood $b \in C$ that we can view as a cone on ∂C with apex b. If b is on the interior of B this would be a small disk around b, and if b is on the boundary it would be a semi-disk (see below).



Thus we can view this as a homotopy H between the inclusion map $\partial C \to B$ and the constant map at b. We may think of this as 2 homotopies,

$$T, D : \partial C \times [0,1] \to B \times B$$

Where $T(c,t) = (H(c,t), b)$ and $D(c,t) = (b, H(c,t))$. We may consider Z as a space over $B \times B$ via the maps (σ, τ). By lemma 4.2, σ and τ are Serre microfibrations, therefore their product is as well. We have an initial lift of both homotopies T and D given by $\iota(f(b))$, thus the both lift in neighborhoods of 0, which correspond to open neighborhoods of b (call these lifts $\tilde{T}$ and $\tilde{D}$). Taking the intersection of these neighborhoods, we obtain a neighborhood U_b in which we have the following data for each $c \in U_b$ by construction:

1. A morphism $\tilde{T}(c) \in Z$ with domain $f(c)$ and target $f(b)$.
2. A morphism $\tilde{D}(c) \in Z$ with domain $f(b)$ and target $f(c)$.

Using the homotopy extension lifting property, one can construct homotopies between the map given by $c \mapsto \tilde{T}(f(c)) \circ \tilde{D}(f(c))$ and the constant map at $\iota(f(b))$, and the maps $c \mapsto \tilde{D}(f(c)) \circ \tilde{T}(f(c))$ and $c \mapsto \iota(f(c))$. Thus on our neighborhood of U_b we can construct maps,

$$\tilde{D}(c) : p^{-1}(b) \to p^{-1}(c)$$

$$f \mapsto \tilde{D}(c) * f$$

$$\tilde{T}(c) p^{-1}(c) \to p^{-1}(b)$$

$$f \mapsto \tilde{T} * f$$

that are continuous in choice of c, and furthermore, the above homotopies tell us that we have homotopies $\tilde{D}(c) \circ \tilde{T}(c) \sim \iota(f(c)) \sim Id_{p^{-1}(c)}$ and $\tilde{T}(c) \circ \tilde{D}(c) \sim \iota(f(b)) \sim Id_{p^{-1}(b)}$ and they are continuous in choice of c as well.

Therefore we have exhibited that our space is weakly fiber homotopy trivial. This tells us it is a weak Serre fibration, and because it is a Serre microfibration as well, we obtain a lift of our map. □

Corollary 2.4. *Suppose that L_{i+1} is obtained from L_i by attaching a handle H of index $k \geq 1$. We have $A^k \times \mathbb{D}^{n-k} \subset H$. Let $L = L_i \cup A^k \times \mathbb{D}^{n-k}$. Then the following pullback square,*

$$\begin{array}{ccc} \boldsymbol{(f)imm}(L \cup H, N) & \longrightarrow & \boldsymbol{(f)imm}(L, N) \\ \downarrow & & \downarrow \\ \boldsymbol{(f)imm}(H, N) & \longrightarrow & \boldsymbol{(f)imm}(L \cap H, N) \end{array}$$

is a weak homotopy square.

Proof. The square is clearly a pullback square, and by lemma 4.1, the bottom arrow is a Serre fibration, thus it is a weak homotopy pullback square. □

2.4 The Inductive Step

Now to finish the proof of theorem 2.2, we are going to induct on the number of handles in the handle decomposition of M.

If M admits a handle decomposition with only a single handle, M must be diffeomorphic to a $\mathbb{D}^n$. Therefore we are done by corollary 2.1. For the inductive step we will also need to know the theorem for $A^k \times \mathbb{D}^{n-k}$. This requires its own inductive argument, but for now we will assume it.

Now assume the theorem for any M which admits a handle decomposition $\emptyset = L_0 \subset L_1 \subset ... \subset L_i = M$, and suppose we have some M which admits a handle decomposition $\emptyset = L_0 \subset L_1 \subset ... \subset L_{i+1} = M$. Let H be the last handle in the decomposition, $A^k \times \mathbb{D}^{n-k} \subset H$ as described before and $L = L_i \cup A^k \times \mathbb{D}^{n-k}$. Now we have the following diagram,

$$
\begin{tikzcd}
\mathbf{imm}(L \cup H, N) \arrow[r] \arrow[d] \arrow[dd, bend right=60] & \mathbf{imm}(L, N) \arrow[d] \arrow[dd, bend left=60] \\
\mathbf{imm}(H, N) \arrow[r] \arrow[dd, bend right=60] & \mathbf{imm}(L \cap H, N) \arrow[dd, bend left=60] \\
\mathbf{fimm}(L \cup H, N) \arrow[r] \arrow[d] & \mathbf{fimm}(L, N) \arrow[d] \\
\mathbf{fimm}(H, N) \arrow[r] & \mathbf{fimm}(L \cap H, N)
\end{tikzcd}
$$

By corollary 2.3 both the top and bottom square are weak homotopy pullback squares. First, note that $L \cap H = A^k \times \mathbb{D}^{n-k}$, so by section 2.3, we know that both the top and bottom squares are weak homotopy pullback squares. Second, notice that L is diffeomorphic to L_i, since $S^k \times \mathbb{D}^{n-k}$ is a retract of $A^k \times \mathbb{D}^{n-k}$, therefore $\mathbf{imm}(L, N) \to \mathbf{fimm}(L, N)$ is a weak homotopy equivalence by the inductive hypothesis. By corollary 2.1 and our assumption for $A^k \times \mathbb{D}^{n-k}$, $\mathbf{imm}(H, N) \to \mathbf{fimm}(H, N)$, and $\mathbf{imm}(L \cap H, N) \to \mathbf{fimm}(L \cap H, N)$ are weak homotopy equivalences as well. Therefore $\mathbf{imm}(L \cup H, N) \to \mathbf{fimm}(L \cup H, N)$ is a weak homotopy equivalence.

Now we finish by proving the theorem for $A^k \times \mathbb{D}^{n-k}$. First, I claim that $A^k \times \mathbb{D}^{n-k}$ has a handle decomposition consisting of only 2 handles. This follows from the fact that we can write, $A^k \times \mathbb{D}^{n-k} \cong S^{k-1} \times [0,1] \times \mathbb{D}^{n-k}$. S^{k-1} has a handle decomposition of 2 handles given by gluing the 2 hemispheres $\mathbb{D}^{k-1}_+$ and $\mathbb{D}^{k-1}_-$, so we can extend this to a handle decomposition, $A^k \times \mathbb{D}^{n-k} \cong \mathbb{D}^{k-1}_+ \times \mathbb{D}^{n-k+1} \times \mathbb{D}^{k-1}_- \times \mathbb{D}^{n-k+1}$, which has 1 handle of index 0 and 1 handle of index $k-1$. We prove the theorem for $A^k \times \mathbb{D}^{n-k}$ by induction on k. For the base case $k = 1$, we are in the case of the disjoint union of 2 n-disks, so the theorem holds. Assume now that we have proven the theorem for $k-1$. Writing down the diagram as above we have,

$$
\begin{tikzcd}
\mathbf{imm}(A^k \times \mathbb{D}^{n-1}, N) \arrow[r] \arrow[d] \arrow[dd, bend right=60] & \mathbf{imm}(\mathbb{D}^n, N) \arrow[d] \arrow[dd, bend left=60] \\
\mathbf{imm}(\mathbb{D}^{k-1} \times \mathbb{D}^{n-k+1}, N) \arrow[r] \arrow[dd, bend right=60] & \mathbf{imm}(A^{k-1} \times \mathbb{D}^{n-k+1}, N) \arrow[dd, bend left=60] \\
\mathbf{fimm}(A^k \times \mathbb{D}^{n-k}, N) \arrow[r] \arrow[d] & \mathbf{fimm}(\mathbb{D}^n, N) \arrow[d] \\
\mathbf{fimm}(\mathbb{D}^{k-1} \times \mathbb{D}^{n-k+1}, N) \arrow[r] & \mathbf{fimm}(A^{k-1} \times \mathbb{D}^{n-k+1}, N)
\end{tikzcd}
$$

By the inductive hypothesis and corollary 2.1, we know that 3 out of the 4 maps are weak homotopy equivalences, thus the fourth is as well.

3 Computing Immersions of S^1 in the Plane

The most important application of the Smale-Hirsch theorem is that it allows us to compute spaces of immersions up to regular homotopy much easier, as we have far more tools to compute $\pi_0(\mathbf{fimm}(M,N))$ than $\pi_0(\mathbf{imm}(M,N))$. As an example, in this section we will apply the theorem to compute the space of immersions of S^1 to the plane up to regular homotopy.

First note that the projection map $p : \mathbf{fimm}(M,N) \to \mathbf{Map}(M,N)$ is a Serre fibration (I will not prove this here but it can easily be checked using the same methods as in the proof of lemma 5.1 for the space of formal immersions). I am going to assume the following theorem,

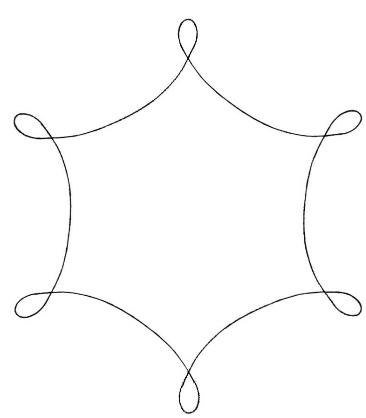
Theorem 3.1. *For $p : E \to B$ a Serre fibration, $b_0 \in B$, $F = p^{-1}(b_0)$ and $x_0 \in F$, there exists a long exact sequence of homotopy groups,*

$$\ldots \to \pi_k(F, x_0) \to \pi_k(E, x_0) \to \pi_k(B, b_0) \to \pi_{k-1}(F, x_0) \to \ldots$$

A proof of this long exact sequence can be found in Allen Hatcher's *Algebraic Topology* (Theorem 4.4 on page 376) [4]. Note that the space $\mathbf{Map}(S^1, \mathbb{R}^2)$ is contractible, thus if we let f be the constant map at $\{0\}$, and $F = p^{-1}(f)$, this long exact sequence tells us that $\pi_0(\mathbf{fimm}(S^1, \mathbb{R}^2)) \cong \pi_0(F)$.

Thus we have now reduced the computation to computing π_0 of the space of vector bundle monomorphisms $TS^1 \to f^*(T\mathbb{R}^2)$ for any choice of f, so we will let $f = Id$. $TS^1 \cong S^1 \times \mathbb{R}$ and $Id^*(T\mathbb{R}^2) \cong S^1 \times \mathbb{R}^2$. Thus this is equivalent to the path space of vector bundle monomorphisms $S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}^2$. The data of such a monomorphism is equivalent to the data of a choice of linear injection $\mathbb{R} \to \mathbb{R}^2$ for every $x \in S^1$ that varies continuously in x. The space of linear injections $\mathbb{R} \to \mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2 - \{0\}$, thus this space is equivalent to $L(\mathbb{R}^2 - \{0\})$, the free loop space on $\mathbb{R}^2 - \{0\}$. $\pi_0(L(\mathbb{R}^2 - \{0\})$ is the space of maps $S^1 \to \mathbb{R}^2 - \{0\}$ up to homotopy, and is therefore $\mathbb{Z}$. Thus, there is exactly one immersion $S^1 \to \mathbb{R}^2$ up to regular homotopy for every $n \in \mathbb{Z}$.

The geometric intuition for this result is the winding number of the immersion. For instance, the immersion assigned to 6 is depicted by the following image,



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