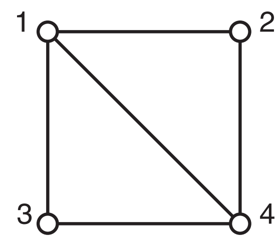
Practice Exam 1

[**1**] Solve the following system of equations:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

[**2**] Compute a matrix giving the number of walks of length 4 between pairs of vertices of the following graph:



[**3**] Express the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

[**4**] Compute the determinant of the following 4×4 matrix:

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 1 & \lambda & 1 & 0 \\ 0 & 1 & \lambda & 1 \\ 0 & 0 & 1 & \lambda \end{bmatrix}$$

What can you say about the determinant of the $n \times n$ matrix with the same pattern?

[**5**] Use Cramer's rule to give a formula for w in the solution to the following system of equations:

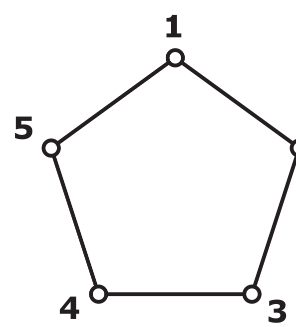
$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Exam 1

[**1**] Solve the following system of equations:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

[**2**] Compute matrices giving the number of walks of lengths 1, 2, and 3 between pairs of vertices of the following graph:



[**3**] Express the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

[**4**] Compute the determinant of the following 4×4 matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 3 & 0 & 3 & 3 \\ 0 & 4 & 4 & 4 \end{bmatrix}$$

What can you say about the determinant of the $n \times n$ matrix with the same pattern?

[**5**] Use Cramer's rule to give a formula for the solution to the following system of equations:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \\ 2c \end{bmatrix}$$

Practice Exam 2

[1] Let P be the set of all polynomials $f(x)$, and let Q be the subset of P consisting of all polynomials $f(x)$ so $f(0) = f(1) = 0$. Show that Q is a subspace of P.

[2] Let A be the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Compute the row space and column space of A.

[3] The four vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

span a subspace V of $\mathbb{R}^3$, but are not a basis for V. Choose a subset of $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4\}$ which forms a basis for V. Extend this basis for V to a basis for $\mathbb{R}^3$.

[4] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which rotates one half turn around the axis given by the vector $(1, 1, 1)$. Find a matrix A representing L with respect to the standard basis

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Choose a new basis $\{\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ which makes L easier to describe, and find a matrix B representing L with respect to this new basis.

[5] Let $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ and $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ be ordered bases for $\mathbb{R}^2$, and let L be the linear transformation represented by the matrix A with respect to $\{\mathbf{e}_1,\ \mathbf{e}_2\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}.$$

Find the transition matrix S corresponding to the change of basis from $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$. Find a matrix B representing L with respect to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$.

Exam 2

[**1**] Let P be the set of all degree ≤ 4 polynomials in one variable x with real coefficients. Let Q be the subset of P consisting of all odd polynomials, i.e. all polynomials $f(x)$ so $f(-x) = -f(x)$. Show that Q is a subspace of P. Choose a basis for Q. Extend this basis for Q to a basis for P.

[**2**] Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Compute the row space and column space of A.

[**3**] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which reflects through the plane P defined by $x + y + z = 0$. In other words, if $\mathbf{u}$ is a vector lying in the plane P, and $\mathbf{v}$ is a vector perpendicular to the plane P, then $L(\mathbf{u} + \mathbf{v}) = \mathbf{u} - \mathbf{v}$. Choose a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, and find a matrix A representing L with respect to this basis.

[**4**] Let $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ and $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ be ordered bases for $\mathbb{R}^2$, and let L be the linear transformation represented by the matrix A with respect to $\{\mathbf{e}_1,\ \mathbf{e}_2\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}.$$

Find the transition matrix S corresponding to the change of basis from $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$. Find a matrix B representing L with respect to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$.

[**5**] Let $\{\mathbf{u}_1,\ \mathbf{u}_2\}$, $\{\mathbf{v}_1,\ \mathbf{v}_2\}$, and $\{\mathbf{w}_1,\ \mathbf{w}_2\}$ be ordered bases for $\mathbb{R}^2$. If

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

is the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1,\ \mathbf{u}_2\}$ to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$, and

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

is the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1,\ \mathbf{u}_2\}$ to $\{\mathbf{w}_1,\ \mathbf{w}_2\}$, express $\mathbf{v}_1$ and $\mathbf{v}_2$ in terms of $\mathbf{w}_1$ and $\mathbf{w}_2$.

Additional Practice Problems for Final

[**1**] By least squares, find the equation of the form $y = ax + b$ which best fits the data $(x_1, y_1) = (0, 1)$, $(x_2, y_2) = (1, 1)$, $(x_3, y_3) = (2, -1)$.

[**2**] Find (s, t) so $\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$ is as close as possible to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

[**3**] Find an orthogonal basis for the subspace $w + x + y + z = 0$ of $\mathbb{R}^4$.

[**4**] Let A be the matrix

$$A = \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}.$$

Find a basis of eigenvectors and eigenvalues for A. Find the matrix exponential e^A.

[**5**] Find a matrix A in standard coordinates having eigenvectors $\mathbf{v}_1 = (1, 1)$, $\mathbf{v}_2 = (1, 2)$ with corresponding eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$.

[**6**] Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Find an orthogonal basis in which A is diagonal.

Final Exam

Linear Algebra, Dave Bayer, December 16, 1999

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Only work which can be found in this way will be graded.

Please do not use calculators or decimal notation.

[1] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects onto the line $(1,1,1)$. In other words, if $\mathbf{u}$ is a vector in $\mathbb{R}^3$, then $L(\mathbf{u})$ is the projection of $\mathbf{u}$ onto the vector $(1,1,1)$. Choose a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, and find a matrix A representing L with respect to this basis.

[2] Compute the determinant of the following 4×4 matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

What can you say about the determinant of the $n \times n$ matrix with the same pattern?

[3] By least squares, find the equation of the form $y = ax + b$ which best fits the data $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (1,1)$, $(x_3, y_3) = (3,1)$.

[4] Find (s,t) so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$ is as close as possible to $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

[5] Find an orthogonal basis for the subspace $w + 2x + 3y + 4z = 0$ of $\mathbb{R}^4$.

[6] Let A be the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Find a basis of eigenvectors and eigenvalues for A. Find the matrix exponential e^A.

[7] Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Find an orthogonal basis in which A is diagonal.

[8] Find a matrix A so the substitution

$$\begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} s \\ t \end{bmatrix}$$

transforms the quadratic form $x^2 + 4xy + y^2$ into the quadratic form $s^2 - t^2$.

Final Exam

Linear Algebra, Dave Bayer, December 16, 1999

Name: SOLUTIONS (done under exam conditions)

ID: ____________ School: ____________

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	
[5] (5 pts)	**[6] (5 pts)**	**[7] (5 pts)**	**[8] (5 pts)**	**TOTAL**

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Only work which can be found in this way will be graded.

Please do not use calculators or decimal notation.

[1] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects onto the line $(1,1,1)$. In other words, if $\mathbf{u}$ is a vector in $\mathbb{R}^3$, then $L(\mathbf{u})$ is the projection of $\mathbf{u}$ onto the vector $(1,1,1)$. Choose a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, and find a matrix A representing L with respect to this basis.

first solution: Choose standard basis $\{(1,0,0),(0,1,0),(0,0,1)\} = E$

proj $(1,0,0)$ onto $(1,1,1) = \left[\dfrac{(1,0,0)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}\right](1,1,1) = \frac{1}{3}(1,1,1)$

proj $(0,1,0)$ onto $(1,1,1) = \frac{1}{3}(1,1,1)$ by symmetry

" $(0,0,1)$ " " " " " " "

So columns of A are each $(1/3, 1/3, 1/3)$

$$A = \underset{E \leftarrow E}{\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}} \quad //$$

over for 2nd solution

Continued on page: 2

Problem: 1

second solution: L takes any vector $\perp$ to (1,1,1) to 0

$\Leftrightarrow$ $\lambda = 0$ has eigenspace $\perp$ to (1,1,1)

L takes (1,1,1) to itself

$\Leftrightarrow$ $\lambda = 1$ has eigenspace spanned by (1,1,1)

so choose eigenvector basis $\{ \underbrace{(1,-1,0), (0,1,-1)}_{\substack{\perp \text{ to } (1,1,1) \\ \lambda = 0}}, \underbrace{(1,1,1)}_{\lambda=1} \} = V$

In this basis we know A is diagonal w/ eigenvalue entries:

$$A = \underset{V \leftarrow V}{\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}} \quad //$$

check these answers against each other: eyeball the inverse **

$$\underset{E \leftarrow E}{\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}} \overset{?}{=} \underset{E \leftarrow V}{\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}} \underbrace{\underset{V \leftarrow V}{\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}} \underset{V \leftarrow E}{\begin{bmatrix} +2 & -1 & -1 \\ +1 & +1 & -2 \\ 1 & 1 & 1 \end{bmatrix}} / 3}_{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} / 3}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} / 3 \quad \checkmark$$

huh? bet there's a 1/3 in inverse! **

on other hand, I'm only using row 3 of inverse, why bother computing other rows??? too late!

how'd I eyeball inverse?

$$\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}$$

inverse

by looking for rows that dotted right with cols of orig matrix. (I liked its columns best...)

Continued on page: ______

[2] Compute the determinant of the following 4×4 matrix:

$$\begin{array}{c} + \\ - \\ + \\ - \end{array} \downarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$ (expand down 1^{st} column)

What can you say about the determinant of the $n \times n$ matrix with the same pattern?

Let $f(n) =$ det of $n \times n$ matrix with same pattern.

$$f(4) = +2 \begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \qquad \left(\begin{vmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \right)$$

$$= +2f(3) \cancel{+} - f(2)$$

$$f(3) = \begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} = +2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - |2|$$

$$= 2f(2) - f(1)$$

$$f(2) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = \cancel{5}\,3 \qquad \text{so } f(3) = 2 \cdot 3 - 2 = 4 \ ??$$

$$f(1) = 2$$

<u>check</u>: $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix}$

$-2 + 8 + -2 = 4$ ✓

$\xrightarrow{\quad}$ middle row to be different! $- + -$

$$f(4) = 2f(3) - f(2)$$
$$= 2 \cdot 4 - 3 = 5 \quad \text{hmm?}$$

guessing here that $f(n) = n+1$. True for $n = 1, 2, 3, 4$

$$\underbrace{f(n+1)}_{n+2} \stackrel{?}{=} \underbrace{2f(n)}_{2(n+1)} - \underbrace{f(n-1)}_{n}$$ ✓ proved by induction!!

$$\boxed{\text{pattern is } n \times n \text{ determinant } = n+1}$$

over for another try...

Continued on page: 4

Problem: 2

how about instead row reducing, tracking determinant?

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & -1/2 & & \\ 0 & 3/2 & -1 & \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = 2\cdot 3/2 \begin{bmatrix} 1 & -1/2 & & \\ 0 & 1 & -2/3 & \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$= 2\cdot 3/2 \cdot 4/3 \begin{bmatrix} 1 & -1/2 & & \\ 0 & 1 & -2/3 & \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = 2\cdot 3/2 \cdot 4/3 \cdot 5/4 \underbrace{\begin{bmatrix} 1 & -1/2 & & \\ & 1 & -2/3 & \\ & & 1 & -3/4 \\ & & & 1 \end{bmatrix}}_{\text{upper triangular, } \det = 1}$$

so $\det = 2 \cdot 3/2 \cdot 4/3 \cdot 5/4 = 5$.

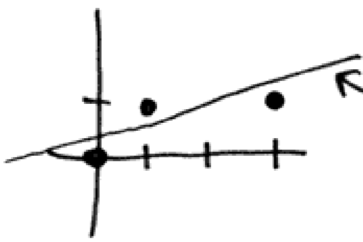
For $n \times n$ case same pattern, $\det = 2 \cdot 3/2 \cdot 4/3 \cdot \ldots \cdot \frac{n+1}{n} = \boxed{n+1}$

so answer checks ✓ //

Continued on page: ______

[3] By least squares, find the equation of the form $y = ax + b$ which best fits the data $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (1, 1)$, $(x_3, y_3) = (3, 1)$.

take a look:



want to solve $\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

or for us, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Overdetermined "$Ax=b$"

so "$A^TAx = A^Tb$" instead:

$$\underbrace{\begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$\det \begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix} = 30 - 16 = 14$

$\begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -4 \\ -4 & 10 \end{bmatrix} / 14$

so $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -4 & 10 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} / 14 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} / 14 = \begin{bmatrix} 2/7 \\ 2/7 \end{bmatrix}$

check: $\begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} / 7 = \begin{bmatrix} 28 \\ 14 \end{bmatrix} / 7 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ ✓

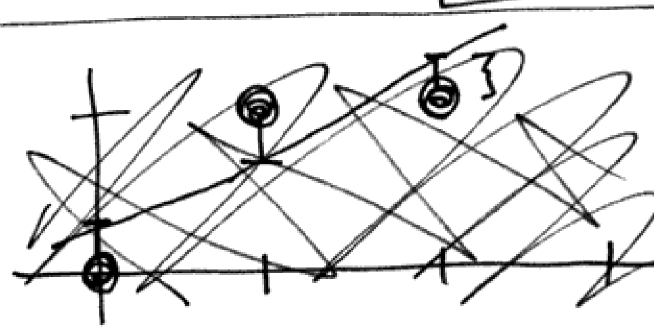
so $a = 2/7 \quad b = 2/7$

$ax + b = \boxed{2/7\, x + 2/7}$

---

check

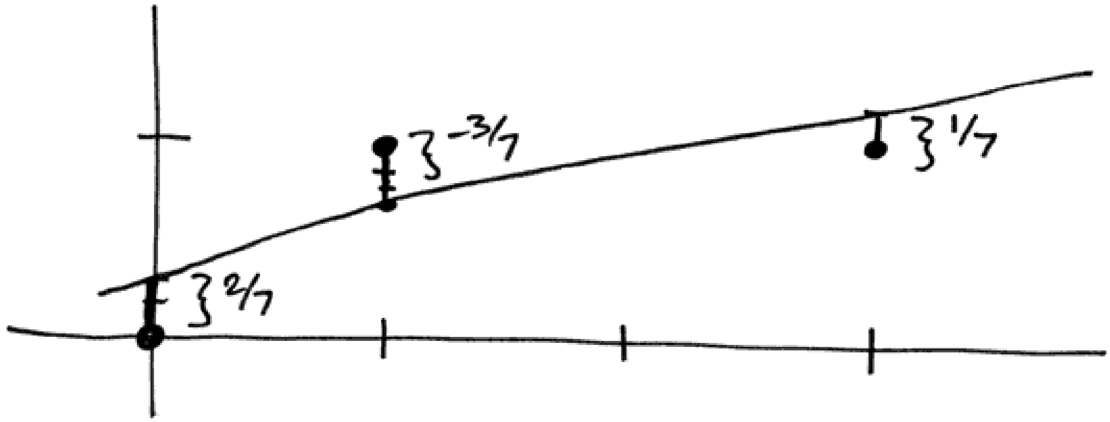
x	y	2/7(x+1)	error
0	0	2/7	2/7
1	1	4/7	-3/7
3	1	8/7	1/7



—over—

Continued on page: 6

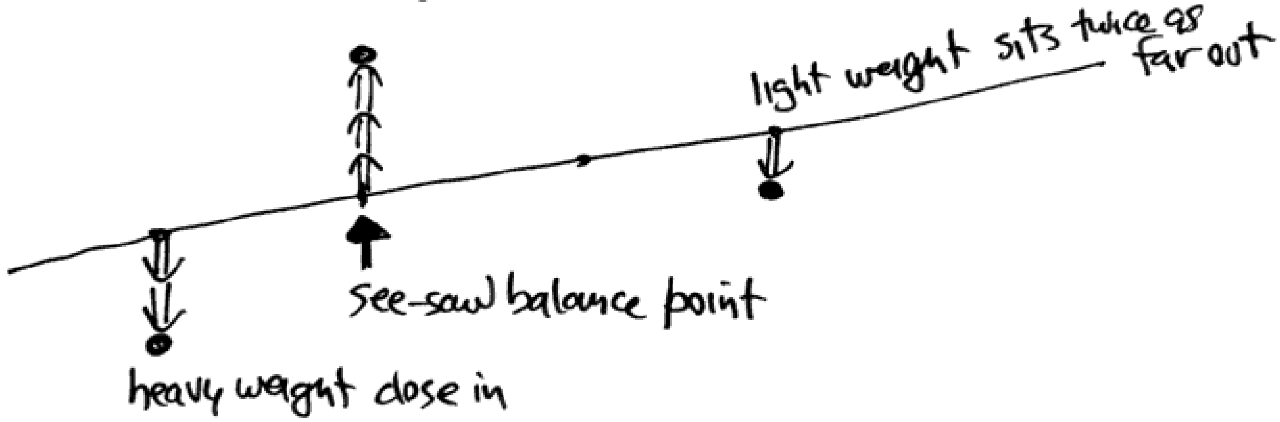
Problem: 3



up and down pulls balance: $2/7 - 3/7 + 1/7 = 0$

$1/7$ down at $x=3$ has twice leverage of $2/7$ down at $x=0$, around pivot of $x=1$, so line doesn't want to twist, either.

looks good //



Continued on page: ______

[4] Find (s,t) so $\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}$ is as close as possible to $\begin{bmatrix}1\\1\\0\end{bmatrix}$.

can't solve $\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$ exactly, Overdetermined "$Ax=b$"
so "$A^TAx = A^Tb$" instead:

$$\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}= \cancel{\quad} \begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

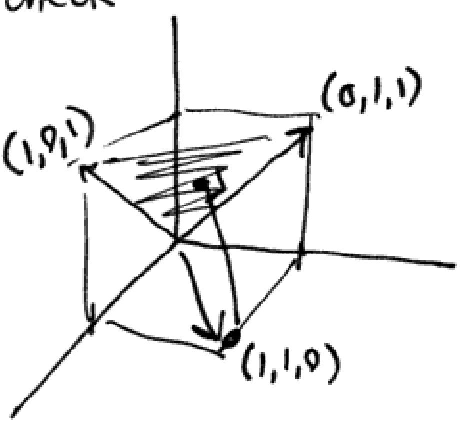
(oops!)

$$\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \qquad \begin{bmatrix}2&1\\1&2\end{bmatrix}^{-1}=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}/3 \quad ✓$$

$$\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}/3\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}/3=\begin{bmatrix}1/3\\1/3\end{bmatrix} \qquad \begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}1/3\\1/3\end{bmatrix}$$

---

check



$$\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\begin{bmatrix}1/3\\1/3\end{bmatrix}=\begin{bmatrix}1/3\\1/3\\2/3\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix}-\begin{bmatrix}1/3\\1/3\\2/3\end{bmatrix}=\begin{bmatrix}2/3\\2/3\\-2/3\end{bmatrix}$$ is $\perp$ to both $(1,0,1)$ $(0,1,1)$

so this is closest point

(vector from nearest point on plane, to $(1,1,0)$, is $\perp$ to plane ✓)

Continued on page: ____

Problem: ______

Continued on page: ______

not ortho<u>normal</u> i.e. lengths don't have to be 1.

[5] Find an orthogonal basis for the subspace $w + 2x + 3y + 4z = 0$ of $\mathbb{R}^4$.

First, equation is $\underline{\underline{1}}$ condition on $\underline{4}$ dimensions, leaving subspace of dim = 3. Expect basis of 3 vectors.

[1] find somehow (eyeball it) 3 indep vectors $\underbrace{\perp \text{ to } (1,2,3,4)}$

(swap and negate, rest zeros) (this is what $w+2x+3y+4z=0$ means)

$v_1 = (2,-1,0,0)$
$v_2 = (0,3,-2,0)$ check $\perp$ $(1,2,3,4)$ ✓
$v_3 = (0,0,4,-3)$

[2] Fix this up to be mutually $\perp$ by Gram-Schmid w/o normalizing to length 1.

$$\boxed{w_1 = v_1 = (2,-1,0,0)}$$

$$\begin{aligned} w_2 = v_2 - (\text{proj } v_2 \text{ onto } w_1) &= v_2 - \left(\frac{v_2\cdot w_1}{w_1\cdot w_1}\right) w_1 \\ &= (0,3,-2,0) - \left(\frac{-3}{5}\right)(2,-1,0,0) \\ &= (0,3,-2,0) + (6/5, -3/5, 0, 0) \\ &= (6/5, 12/5, -2, 0) \\ &\sim (6,12,-10,0) \sim (3,6,-5,0) \end{aligned}$$

check $(3,6,-5,0)\cdot(1,2,3,4) = 0$ ✓
$\cdot (2,-1,0,0) = 0$ ✓

(clean up numbers by changing length for easiest arithmetic.)

$$\boxed{w_2 = (3,6,-5,0)}$$

$$\begin{aligned} w_3 &= v_3 - (\text{proj } v_3 \text{ onto } w_1) - (\text{proj } v_3 \text{ onto } w_2) \\ &= (0,0,4,-3) - \left(\frac{0}{\cancel{}}\right)(\cancel{}) - \left(\frac{-20}{9+36+25}\right)(3,6,-5,0) \\ &= (0,0,4,-3) + \frac{20}{70}(3,6,-5,0) \end{aligned}$$

-over-

Continued on page: 10

Problem: 5 (copy carefully)

$$w_3 = (0,0,4,-3) + \tfrac{2}{7}(3,6,-5,0)$$

$$\sim 7(0,0,4,-3) + 2(3,6,-5,0) \quad \text{(rescale by denom)}$$

$$= (0,0,28,-21) + (6,12,-10,0)$$

$$= (6,12,18,-21)$$

$$\sim (2,4,6,-7) \quad \text{(pull out a 3)}$$

check: $(2,4,6,-7)\cdot(1,2,3,4) = 2+8+18-28 = 0$ ✓

$(2,-1,0,0) = 0$ ✓

$(3,6,-5,0) = 6+24-30 = 0$ ✓

whew !!

$$\boxed{\begin{aligned} w_1 &= (2,-1,0,0) \\ w_2 &= (3,6,-5,0) \\ w_3 &= (2,4,6,-7) \end{aligned}}$$

Note, perhaps easier to choose in different order:

$$v_1 = w_1 = \boxed{(2,-1,0,0)}$$

$$v_2 = w_2 = \boxed{(0,0,4,-3)} \quad \text{(already } \perp\text{)}$$

$$v_3 = (0,3,-2,0)$$

$$w_3 = v_3 - \left(\frac{v_3\cdot w_1}{w_1\cdot w_1}\right)w_1 - \left(\frac{v_3\cdot w_2}{w_2\cdot w_2}\right)w_2$$

$$= (0,3,-2,0) - \left(\frac{-3}{5}\right)(2,-1,0,0) - \left(\frac{-8}{25}\right)(0,0,4,-3)$$

$$\sim 25(0,3,-2,0) + 15(2,-1,0,0) + 8(0,0,4,-3)$$

$$= (0,75,-50,0) + (30,-15,0,0) + (0,0,32,-24)$$

$$= (30,60,-18,-24) \sim \boxed{(5,10,-3,-4)} \quad \text{clearly } \perp \text{ to } w_1, w_2$$

check $(1,2,3,4)\cdot(5,10,-3,-4) = 5+20-9-16 = 0$ ✓

Continued on page:

[6] Let A be the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Find a basis of eigenvectors and eigenvalues for A. Find the matrix exponential e^A.

$$|A-\lambda I| = \lambda^2 - \underbrace{(\text{trace of } A)}_{\substack{(\text{sum of diag}) \\ 3+3}} \lambda + (\det \text{ of } A) = 0$$

$\lambda^2 - 6\lambda + 5 = 0$ factors as $(\lambda-1)(\lambda-5)$

so $\lambda = 1, 5$.

$\boxed{\lambda_1 = 1}$ $A - 1I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ has kernel basis $\boxed{(1,-1) = v_1}$

$\boxed{\lambda_2 = 5}$ $A - 5I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ has kernel basis $\boxed{(1,1) = v_2}$

check: A symmetric, and $v_1 \perp v_2$ as expected.

check: $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓

$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

so we have E = standard basis, V = eigen~~vec~~ basis $v_1 = (1,-1)$ $v_2 = (1,1)$

(check inverse ✓)

$$A = \underset{E \leftarrow E}{\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}} = \underbrace{\underset{E \leftarrow V}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}}_{\substack{\text{columns} \\ \text{are } v_1, v_2}} \underset{V \leftarrow V}{\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}} \underbrace{\underset{V \leftarrow E}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}/2}}_{\text{inverse matrix}}$$

check:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ 5 & 5 \end{bmatrix}/2$$

$$\begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}/2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} ✓$$

Continued on page: 12

—over—

Problem: 6

So we can compute e^A by this same change of coords:

$$\underset{E\leftarrow E}{e^A} = \underset{E\leftarrow V}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}} \; \underset{V\leftarrow V}{e^{\begin{bmatrix} 1 & \\ & s \end{bmatrix}}} \; \underset{V\leftarrow E}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}} / 2$$

$$\boxed{e^A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e^s \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} / 2}$$

(OK to leave in this form <u>if</u> correct)

or multiplied out,

$$e^A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e & -e \\ e^s & e^s \end{bmatrix} / 2$$

$$\boxed{e^A = \frac{1}{2} \begin{bmatrix} e+e^s & -e+e^s \\ -e+e^s & e+e^s \end{bmatrix}}$$

Argh! e^A should be symmetric, I just caught a sign error in copying from previous page, that I never would have caught without multiplying out and staring at answer to see if I believed it. So multiplying out isn't <u>required</u> but is certainly <u>safer</u> !!

Continued on page: ______

[7] Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Find an orthogonal basis in which A is diagonal.

First, short direct attack by cleverness:

$$A = \underbrace{\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}}_{\substack{\text{stretches}\\ \text{uniformly by } 2\\ \lambda = 2,2,2}} + \underbrace{\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}}_{\substack{\text{swaps } 1^{st} \text{ and } 3^{rd}\\ \text{coords so:}}}$$

$$(1,0,1) \longmapsto (1,0,1) \quad \lambda = 1$$
$$(1,0,-1) \longmapsto (-1,0,1) \quad \lambda = -1$$
$$(0,1,0) \longmapsto (0,1,0) \quad \lambda = 1$$

so combined effect is

$$(1,0,1) \xmapsto{A} 2(1,0,1) + (1,0,1) = 3(1,0,1) \qquad \lambda = 2+1 = 3$$
$$(0,1,0) \xmapsto{A} 2(0,1,0) + (0,1,0) = 3(0,1,0) \qquad \lambda = 2+1 = 3$$
$$(1,0,-1) \xmapsto{A} 2(1,0,-1) + (-1,0,1) = (1,0,-1) \qquad \lambda = 2-1 = 1$$

And this basis <u>is</u> orthogonal:

$$(1,0,1)\cdot(1,0,-1) = 0 \ ✓$$
$$(1,0,1)\cdot(0,1,0) = 0 \ ✓$$
$$(1,0,-1)\cdot(0,1,0) = 0 \ ✓$$

Eigenvector $\perp$ basis (expected because A symmetric):

$$\boxed{\begin{array}{ll} V_1 = (1,0,1) & \lambda_1 = 3 \\ V_2 = (1,0,-1) & \lambda_2 = 1 \\ V_3 = (0,1,0) & \lambda_3 = 3 \end{array}}$$

caught a mistake !!!!!

E = usual coords
V = eigenvector basis

<u>check</u> that A is diagonal in this basis:

$$\underset{E \leftarrow E}{A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}} \overset{?}{=} \underbrace{\underset{E \leftarrow V}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}}}_{\substack{\text{cols are}\\ V_1, V_2, V_3}} \underset{V \leftarrow V}{\begin{bmatrix} 3 & & \\ & 1 & \\ & & 3 \end{bmatrix}} \underbrace{\underset{V \leftarrow E}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix} \Big/ 2}}_{\substack{\text{inverse}\\ \text{(eyeball it)}}}$$

(check inverse: ✓)

(eyeball it:)
$$\begin{bmatrix} * & * & 0 \\ 0 & 0 & 1 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} * & 0 & * \\ * & 0 & * \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

is just a 2×2 inverse hiding in a 3×3 matrix

and $\begin{bmatrix} * & * \\ * & * \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Big/ 2$ ✓

Continued on page: 14

—over—

Problem: 7

(copy carefully this time!) (check copying ☑) (and check copying on other problems ☑)
(Did you just check the box w/o doing it?!!!)

$$\begin{bmatrix}2&0&1\\0&3&0\\1&0&2\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}1&1&0\\0&0&1\\1&-1&0\end{bmatrix}\underbrace{\begin{bmatrix}3&&\\&\cancel{3}1&\\&&\cancel{1}3\end{bmatrix}\begin{bmatrix}1&0&1\\1&0&-1\\0&2&0\end{bmatrix}/2}_{\begin{bmatrix}3&0&3\\1&0&-1\\0&6&0\end{bmatrix}/2}$$

$$\begin{bmatrix}4&0&2\\0&6&0\\2&0&4\end{bmatrix}/2$$

☑ Whew!!

Argh!!! This isn't coming out. How will I ever get that A+ if I don't hand in a perfect paper?? Where did I go wrong....?? ... Ahh, spoonerized the eigenvectors, fix up, ~~and~~ and go on...

How would I have figured out eigenvectors by "usual" machine?

$$\begin{vmatrix}2-\lambda&0&1\\0&3-\lambda&0\\1&0&2-\lambda\end{vmatrix} = (2-\lambda)\begin{vmatrix}3-\lambda&0\\0&2-\lambda\end{vmatrix} + 1\begin{vmatrix}0&1\\3-\lambda&0\end{vmatrix}$$

$$= (2-\lambda)^2(3-\lambda) - (3-\lambda)$$

$$= [(2-\lambda)^2-1](3-\lambda) = [\lambda^2-4\lambda+3](3-\lambda)$$

$$= -(\lambda-1)(\lambda-3)(\lambda-3)$$

so $\lambda = 1,3,3$

$\lambda=3$: $\begin{bmatrix}2-3&0&1\\0&3-3&0\\1&0&2-3\end{bmatrix} = \begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix}$ has kernel basis $\boxed{(1,0,-1),\ (0,1,0) \quad \lambda=3}$

$\lambda=1$ matrix is symmetric so 3rd eigenvector is $\perp$ to these, i.e. $\boxed{(1,0,1) \quad \lambda=1}$

Continued on page: ______

[8] Find a matrix A so the substitution

$$\begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} s \\ t \end{bmatrix}$$

transforms the quadratic form $x^2 + 4xy + y^2$ into the quadratic form $s^2 - t^2$.

(Unfair question, did we do anything like this? We did discuss quadratic forms in several classes...)

$$x^2 + 4xy + y^2 = [x \ y] \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so if I can understand good coords for matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ I'll understand problem.

Is this an eigenvector problem in disguise?

$|A - \lambda I| = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$ so $\lambda = -1, 3$

$\lambda = -1$: $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ has kernel $\boxed{(1,-1) \ \lambda = -1}$

symmetric so eigenvectors are $\perp$: $\boxed{(1,+1) \ \lambda = 3}$

check $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ +1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓

(I had written $(1,-1)$ without thinking, just caught it by checking. See the sign change?)

so $(1,1)$ and $(1,-1)$ is a good coord system.

Let's try plugging in $\underbrace{x = s+t \quad y = s-t}$ (inspired by $(1,1), (1,-1)$) and see what happens, fix later.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} \quad \text{i.e. } A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{aligned} x^2 + 4xy + y^2 &= (s+t)^2 + 4(s+t)(s-t) + (s-t)^2 \\ &= s^2 + 2st + t^2 + 4s^2 - 4t^2 + s^2 - 2st + t^2 \\ &= \boxed{6s^2 - 2t^2} \end{aligned}$$

Wow, pretty close.
I see the 2 (from Taylor's thm ???)
the $-1, 3$ from eigenvalues.

There's probably some theory for rescaling, but let's just wing it.

Continued on page: 16

—over—

Problem: 8

I have $6s^2 - 2t^2$, want $s^2 - t^2$

replace s by $\frac{1}{\sqrt{6}}s$, t by $\frac{1}{\sqrt{2}}t$ would work.

Let's find this out by general substitution

$$\underbrace{x = as+bt \quad y = as-bt}_{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ a & -b \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}}$$

(I guess in that general theory, it's the columns of A that matter. Pretty typical...)

$$x^2 + 4xy + y^2 = (as+bt)^2 + 4(as+bt)(as-bt) + (as-bt)^2$$
$$= a^2s^2 + 2abst + b^2t^2 + 4a^2s^2 - 4b^2t^2 + a^2s^2 - 2abst + b^2t^2$$
$$= 6a^2s^2 - 2b^2t^2 \underset{\text{want}}{=} s^2 - t^2$$

so $6a^2 = 1$, $2b^2 = 1$ or $a = 1/\sqrt{6}$, $b = 1/\sqrt{2}$ (not cleaning this up drove my high school teachers crazy)

answer: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}$

so what's the general theory, and how do eigenvectors enter in?

$$\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \cdot \underbrace{3\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}}_{\text{because eigenvector}} = 3\left(\tfrac{1}{6}+\tfrac{1}{6}\right) = +1$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \left[1/\sqrt{2} \; -1/\sqrt{2}\right) \cdot (-1)\left(1/\sqrt{2}, -1/\sqrt{2}\right) = -1\left(\tfrac{1}{2}+\tfrac{1}{2}\right) = -1$$

---

if $Av = \lambda v$

$v \cdot Av = v \cdot \lambda v = \lambda\, v \cdot v$ so we need to rescale v so $v \cdot v = \left|\frac{1}{\lambda}\right|$.

Nice theory for next time...

Continued on page: //

(20 minutes left to look over test again to nail that A+ ?)