

Exercise: How many distinct ways are there of marking 3 circles on a triangular board of 10 circles, *up to symmetry*?

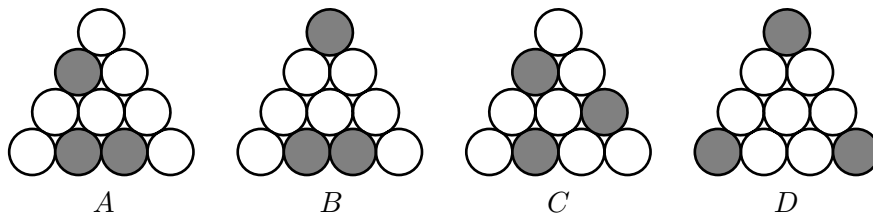


Figure 1

For example, pattern *A* in Figure 1 is totally asymmetric, so there are a total of 6 patterns which look like *A* under the action of the triangle group. We only want to count such a pattern once.

Pattern *B* has a flip symmetry, so there are two other patterns which look like it, obtained by rotation. Pattern *C* has rotational symmetry, so there is one other pattern which looks like it, obtained by flipping. Pattern *D* is totally symmetric. We only want to count each of these patterns once.

Let  $x_A, x_B, x_C, x_D$  be the number of patterns of symmetry types *A, B, C,* and *D,* counted up to symmetry. It is hard to compute these quantities directly; it is easier to count patterns having *at least* a certain kind of symmetry, without worrying about matching up patterns which are really the same. We then figure out the coefficients of  $x_A, x_B, x_C, x_D$  in these easier counts, in order to solve for  $x_A + x_B + x_C + x_D$ .

We know that there are three *conjugate* flavors of flip symmetry, corresponding to the three possible axis of flipping. It is enough to only consider one of these flavors, only counting patterns having at least, say, vertical axis flip symmetry. Every pattern with flip symmetry has one representative with vertical axis flip symmetry, so we will count all such patterns once.

There is only one flavor of rotational symmetry, because rotations are a *normal* subgroup of the triangle group. Every pattern with rotational symmetry has two representatives, one the flip of the other.

[1] Let  $y_A$  be the number of ways of marking 3 circles on our board, if we don't worry about symmetry. Show that  $y_A = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ .

(Hint: There are 10 choices for where to mark first, and then 9 choices left for where to mark second, and then 8 choices left for where to mark third. However, we don't care in what order we mark a given pattern. How many times does  $10 \cdot 9 \cdot 8$  count each pattern?)

[2] Let  $y_B$  be the number of ways of marking 3 circles on our board, so the resulting pattern is symmetric with respect to flipping across the vertical axis. To get an easy formula, don't worry about also counting patterns which are even more symmetric. Show that  $y_B = 2 \cdot 4$ .

[3] Let  $y_C$  be the number of ways of marking 3 circles on our board, so the resulting pattern is rotationally symmetric. Don't worry if two patterns in your count are the same after flipping, and don't worry about counting patterns which are even more symmetric. Show that  $y_C = 3$ .

[4] Let  $y_D$  be the total number of ways of marking 3 circles on our board, so the resulting pattern is completely symmetric. Show that  $y_D = 1$ .

[5] Now consider two patterns to be the same if they are the same up to symmetry. Figure out the coefficients of the linear equation

$$y_A = ax_A + bx_B + cx_C + dx_D.$$

In other words, how many times does  $y_A$  count a pattern which is totally asymmetric, like  $A$ ? This is the coefficient  $a$ . How many times do patterns of types  $B$ ,  $C$ , and  $D$  get counted? These are the coefficients  $b$ ,  $c$ , and  $d$ .

[6] Figure out similar linear equations for  $y_B$ ,  $y_C$ , and  $y_D$ . Solve this system of equations to find  $x_A$ ,  $x_B$ ,  $x_C$ , and  $x_D$ . The sum  $x_A + x_B + x_C + x_D$  is the number of distinct ways there are of marking 3 circles on a triangular board of 10 circles, up to symmetry.

This is a tricky problem, until you get the hang of this method. If you can't get your system of equations to work out nicely, list all the possibilities of types  $B$ ,  $C$ , and  $D$  by hand, and substitute these values into your equations to see where you went wrong.

Notice that your reasoning in [5] and [6] has nothing to do with the board size, or the number of marks; it only depends on the structure of the triangle group. We can use the above system of linear equations to solve for  $x_A + x_B + x_C + x_D$  in terms of  $y_A$ ,  $y_B$ ,  $y_C$ , and  $y_D$ . This final equation is useful for other size problems; we only have to recompute  $y_A$ ,  $y_B$ ,  $y_C$ , and  $y_D$ . Try it!

The truly obsessed may want to play with the group of 8 symmetries of the square, and redo this entire problem for marking square boards. Reviewing both problems, can you see a pattern to the coefficients you get in the final equations?

The slickest way of stating this counting technique is known as Polya enumeration.