First midterm

Dave Bayer, Modern Algebra, October 14, 1998

Work as many parts of each problem as you can, while budgeting your time. For a successful exam, it isn't necessary to answer every part of every question.

[1] Let $G = S_3 = \{\, (),\ (1\ 2),\ (1\ 3),\ (2\ 3),\ (1\ 2\ 3),\ (1\ 3\ 2)\,\}$ be the symmetric group of all permutations of $\{\,1,\ 2,\ 3\,\}$, and let H be the subgroup $H = \{\,(\,),\ (1\ 2)\,\} \subset G$.
(a) List the left cosets of H in G.
(b) List the right cosets of H in G.
(c) Is H normal in G?

[2] Let m and n be relatively prime integers, and consider the two groups $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and $H = \mathbb{Z}_{mn}$.
(a) Present each of these groups in terms of generators and relations.
(b) Find an isomorphism between G and H.
(c) Give an example showing what happens when m and n aren't relatively prime.

[3] Let G be the group presented in terms of generators and relations by

$$G \quad = \quad \langle\, a, b \mid a^2 = b^2 = 1,\ bab = aba \,\rangle.$$

Describe G as completely as you can. (Suggestions: How many elements does G have? List representatives for the distinct elements of G. Is G abelian or not? Draw the Cayley graph of G. Recognize G as isomorphic to a familiar group, and give an explicit isomorphism.)

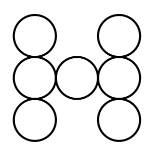

Figure 1

[4] Describe the group G of symmetries of the configuration of cells shown in Figure 1, considering both rotations and flips. How many ways are there of marking two of the cells in Figure 1, up to symmetry? Use Burnside's formula

$$(\#\text{ of patterns up to symmetry}) \quad = \quad \frac{1}{|G|} \sum_{g \in G} (\#\text{ of patterns fixed by } g).$$

[5] The *normalizer* $N(H)$ of a subgroup H of a group G can be defined to be the set

$$N(H) \quad = \quad \{\, g \in G \mid gHg^{-1} = H \,\}.$$

(a) Prove that $N(H)$ is a subgroup of G.
(b) Prove that H is a normal subgroup of $N(H)$.
(c) Suppose that J is another subgroup of G *conjugate* to H: $H \neq J$, but $aHa^{-1} = J$ for some $a \in G$. Describe the set $\{\, g \in G \mid gHg^{-1} = J \,\}$ in terms of a and $N(H)$.
(d) How many subgroups of G are conjugate to H, counting H itself?