First Midterm

Dave Bayer, Modern Algebra, October 6, 1997

[1] Give an example of a group G and a subgroup H, where
(a) H is normal. What is the quotient group G/H?
(b) H is not normal. Show that H is not normal, by finding an element $g \in G$ with the property that the cosets $gH \neq Hg$.

[2] Let G be the group of 2 by 2 matrices whose entries are integers mod 5, and whose determinant is nonzero mod 5. Let H be the subset of G consisting of all matrices whose determinant is 1 mod 5.
(a) How many elements are there in G and in H?
(b) Show that H is a normal subgroup of G.
(c) What familiar group is isomorphic to the quotient group G/H?

[3] Draw the Cayley graph of the group of order 10 defined using generators and relations by
$$G \quad = \quad \langle\, a, b \mid a^5 = 1,\ b^2 = 1,\ ba = ab \,\rangle.$$

What familiar group is isomorphic to G? Give an isomorphism.

[4] The *center* $Z(G)$ of a group G is the set of all elements of G which commute with every element of G:

$$Z(G) \quad = \quad \{\, g \in G \mid gh = hg \text{ for every } h \in G \,\}.$$

(a) Show that $Z(G)$ is a subgroup of G.
(b) Show that $Z(G)$ is in fact a normal subgroup of G.
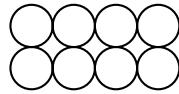

Figure 1

[5] How many ways are there of marking some (or none, or all) of the cells in Figure 1, up to symmetry? Consider two patterns to be the same if one can be obtained from the other by rotating or flipping. Use Burnside's formula

$$(\#\text{ of patterns up to symmetry}) \quad = \quad \frac{1}{|G|} \sum_{g \in G} (\#\text{ of patterns fixed by } g),$$

where G is the group of symmetries of this configuration of cells.