Second homework problems

Dave Bayer, Modern Algebra, September 30, 1998

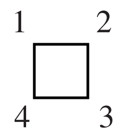

Figure 1

Problems 1 through 5 all concern the group G of symmetries of a square.

[1] The group G of symmetries of the square has 8 elements: the identity, 3 rotations, and 4 flips. G can be generated by two elements: a quarter-turn a, and a flip b. By numbering the square as shown in Figure 1, express the 8 elements of G as permutations on $\{1, 2, 3, 4\}$. This expresses G as a subgroup of the group S_4 of all permutations on $\{1, 2, 3, 4\}$. What permutations represent your choices of a and b?

[2] In terms of generators and relations, G can be written as the group

$$G \quad = \quad \langle\, a, b \mid a^4 = b^2 = 1, \;\; b\,a = a^m\, b \,\rangle$$

for some choice of m. What is m? Listing the elements of G in the form

$$1,\; a,\; a^2,\; a^3,\; b,\; a\,b,\; a^2\,b,\; a^3\,b,$$

describe each of these elements both as symmetries of the square, and as permutations in S_4.

[3] Show that S_4 can be generated by the two permutations $c = (1\ 2)$ and $d = (1\ 2\ 3\ 4)$. How would you demonstrate this to a high school student, using props but no notation?

[4] Decide whether or not G is a normal subgroup of S_4, using the criterion

$$G \subset S_4 \quad \text{is normal} \quad \Longleftrightarrow \quad g\,G\,g^{-1} \;=\; G \quad \text{for all} \quad g \in S_4.$$

Show that it is enough to check that $g\,h\,g^{-1} \in G$ for each generator g of S_4 and each generator h of G. Apply this to the generators c, d of S_4 and a, b of G which you found above, checking

$$c\,a\,c^{-1} \in G\,? \qquad c\,b\,c^{-1} \in G\,? \qquad d\,a\,d^{-1} \in G\,? \qquad d\,b\,d^{-1} \in G\,?$$

[5] Decide whether or not the subgroup $H = \{1, a, a^2, a^3\} \subset G$ generated by a is normal in G, using the same method. Repeat, for the subgroup $H = \{1, b\} \subset G$ generated by b.

The remaining problems are independent of each other.

[6] Let the group G be given in terms of generators and relations as

$$G \quad = \quad \langle\, a,\, b,\, c \mid a\,b\,c = a\,c\,b = 1 \,\rangle.$$

How many distinct elements does G have? Your answer may surprise you. Can you give a simpler presentation of G, perhaps using fewer generators? Do you recognize G?

[7] Let the group G be given in terms of generators and relations as

$$G \quad = \quad \langle\, a,\, b \mid a^3 = b^3 = 1,\;\; b\,a = a^2\,b \,\rangle.$$

How many distinct elements does G have? Your answer may surprise you. Can you give a simpler presentation of G, perhaps using fewer generators?

[8] (challenging) Let p and q be prime numbers. For which integers m, $1 \leq m < p$, does the presentation

$$G \quad = \quad \langle\, a,\, b \mid a^p = b^q = 1,\;\; b\,a = a^m\,b \,\rangle.$$

describe a group with $p\,q$ distinct elements? For these values of m, when is the subgroup generated by a normal? When is the subgroup generated by b normal?