Final Examination

Dave Bayer, Modern Algebra, December 23, 1997

Each problem is worth 5 points for a total of 50 points. Work as much of each problem as you can.

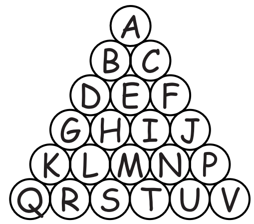

Figure 1

[**1**] Let the dihedral group D_3 of symmetries of the triangle act on the cells shown in Figure 1. For example, a vertical axis flip takes cell **B** to cell **C**, and a clockwise rotation takes cell **B** to cell **P**.

What are the orbits of D_3, acting on the set of cells

$$\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{H}, \mathsf{I}, \mathsf{J}, \mathsf{K}, \mathsf{L}, \mathsf{M}, \mathsf{N}, \mathsf{P}, \mathsf{Q}, \mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V}\}?$$

[**2**] Let a, b be two elements of a group G, and let H be a subgroup of G. Consider the left cosets aH and bH of H in G. Show that if aH and bH have any elements in common, then $aH = bH$.

[**3**] The *center* Z of a group G is the set of elements of G which commute with all elements of G:

$$Z = \{\ g \in G \mid gh = hg \text{ for all } h \in G\ \}.$$

The *centralizer* $Z(x)$ of an element $x \in G$ is the set of elements of G which commute with x:

$$Z(x) = \{\ g \in G \mid gx = xg\ \}.$$

(a) Show that $Z(x)$ is a subgroup of G.

(b) Show that $x \in Z$ if and only if $Z(x) = G$.

[**4**] The centralizer $Z(x)$ of $x \in G$ can also be thought of as the stabilizer of x with respect to conjugation:

$$Z(x) = \{\ g \in G \mid gxg^{-1} = x\ \}.$$

Suppose that $axa^{-1} = y$ for some $a \in G$, so x and y are conjugate elements of G.

(a) Describe the subset $\{\ g \in G \mid gxg^{-1} = y\ \}$ in terms of $Z(x)$ and a.

(b) Show that the number of elements of G conjugate to x is given by the formula $|G|\,/\,|Z(x)|$.

(c) Show that any group of order p^2 is abelian, when p is prime.

[**5**] Let U, V, and W be three subspaces of a finite-dimensional vector space over a field F. Prove that

$$\dim(U + V + W) \ \leq\ \dim(U)\ +\ \dim(V)\ +\ \dim(W).$$

[6] Find the multiplicative inverse of 102 mod 103.

[7] Let

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}.$$

Find a change of basis matrix B so $A = B\,C\,B^{-1}$ where C is in Jordan canonical form. Use B and C to find e^{At}.

[8] Let $G = \{1, a, a^2, b, ab, a^2b\} = \langle\, a, b \mid a^3 = b^2 = 1,\ ba = a^{-1}b\,\rangle$; this is a presentation of the dihedral group D_3. Let $U = \{a, a^2, ab, a^2b\} \subset G$, and let G act on itself by left multiplication.

(a) What is the stabilizer $H = \mathrm{Stab}(U)$ of U?

(b) List the right cosets Ha of H in G.

(c) Express U as a union of right cosets of H, and verify that $|H|$ divides $|U|$.

[9] Let G be a group of order $p^e\,m$, where p is a prime that does not divide m. Prove that G has a subgroup H of order p^e.

[10] Classify the groups of order n, where

(a) n = 33.

(b) n = 39.

(c) n = 49.