Practice problems for second midterm

Dave Bayer, Modern Algebra, November 5, 1997

There will be two review sessions in 528 Mathematics: Friday, November 7 at 2:40pm, and Sunday, November 9 at 2:40pm. (We will move if we need more space.)

[1] Define an abstract field, and a vector space over an abstract field. Define a homomorphism of abstract fields, and a homomorphism of vector spaces.

[2] Let p be a prime, and let $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements.

(a) What is the order of the group of invertible elements of $\mathbf{F}_p$?

(b) Find the multiplicative inverse of 17 mod 31, by either using the extended Euclidean algorithm, or by taking successive powers.

[3] Let p and q be primes, and let $\mathbb{Z}/pq\mathbb{Z}$ be the ring of integers modulo pq.

(a) What is the order of the group of invertible elements of $\mathbb{Z}/pq\mathbb{Z}$?

(b) Find the multiplicative inverse of 17 mod 91, by either using the extended Euclidean algorithm, or by taking successive powers.

[4] Let the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ with respect to the standard basis for $\mathbb{R}^2$.

(a) Find a matrix $B = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, or $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ which is similar to A.

(b) Find a basis v_1, v_2 for $\mathbb{R}^2$ such that the matrix for T with respect to this basis is B.

The following problems each involve working with linearly independent sets, spanning sets, or bases.

[5] Let V be a finite-dimensional vector space over a field F, and suppose that $v_1, \ldots, v_n$ spans V. Prove that $v_1, \ldots, v_n$ contains a basis of V.

[6] Let V be a finite-dimensional vector space over a field F, and suppose that $v_1, \ldots, v_n$ are linearly independent vectors in V. Prove that $v_1, \ldots, v_n$ can be extended to a basis of V.

[7] Let V be a finite-dimensional vector space over a field F, and suppose that $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are two bases of V. Prove that $m = n$.

[8] Let V and W be finite-dimensional vector spaces over a field F, and let $A : V \to W$ be a linear transformation. Prove that

$$\dim(V) \quad = \quad \dim(\ker A) \; + \; \dim(\operatorname{im} A).$$

[9] Let W_1 and W_2 be two subspaces of a finite-dimensional vector space V over a field F. Prove that

$$\dim(W_1) \ + \ \dim(W_2) \quad = \quad \dim(W_1 \cap W_2) \ + \ \dim(W_1 + W_2).$$

[10] Let W_1, W_2 and W_3 be three subspaces of a finite-dimensional vector space V over a field F. Prove that

$$\dim(W_1 + W_2 + W_3) \quad \leq \quad \dim(W_1) \ + \ \dim(W_2) \ + \ \dim(W_3).$$

[11] Let V be a finite-dimensional vector space over a field F, and let $A : V \to V$ be a linear transformation such that $A^2 = 0$. In other words, we have

$$V \xrightarrow{\ A\ } \operatorname{im} A \xrightarrow{\ A\ } 0.$$

(a) Find a basis $v_1, \ldots, v_n$ for V, such that the matrix for A with respect to this basis is block diagonal with blocks $[0]$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Prove that $v_1, \ldots, v_n$ is in fact a basis for V.

(b) Show that $\dim(\operatorname{im} A) \leq \frac{1}{2} \dim(V)$.

[12] Let V be a finite-dimensional vector space over a field F, and let $A : V \to V$ be a linear transformation such that $A^3 = 0$.

(a) Find a basis $v_1, \ldots, v_n$ for V, such that the matrix for A with respect to this basis is block diagonal with blocks $[0]$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Prove that $v_1, \ldots, v_n$ is in fact a basis for V.

(b) What bounds must hold between the dimensions $\dim(V)$, $\dim(\operatorname{im} A)$, and $\dim(\operatorname{im} A^2)$?

[13] Let V be a finite-dimensional vector space over a field F, and let $A : V \to V$ be a linear transformation such that $\dim(\operatorname{im} A) < \dim(V)$, but $\operatorname{im} A^2 = \operatorname{im} A$. In other words, the restriction of the linear transformation $A : \operatorname{im} A \to \operatorname{im} A$ is nonsingular (an isomorphism).

Let $n = \dim(V)$, and $m = \dim(\operatorname{im} A)$. Find a basis $v_1, \ldots, v_m, \ldots, v_n$ for V, such that the matrix for A with respect to this basis has all zero entries outside of an m by m diagonal block corresponding to the subspace $\operatorname{im} A$.