Practice Exam 2

Linear Algebra, Dave Bayer, November 2, 2000

Name: SOLUTIONS

ID: ______ **School:** ______

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Only work which can be found in this way will be graded.

Please do not use calculators or decimal notation.

[**1**] Let

$$\mathbf{v}_1 = (-1, 2, -1, 0), \quad \mathbf{v}_2 = (-1, 1, 1, -1), \quad \mathbf{v}_3 = (0, -1, 2, -1).$$

Find a basis for the subspace $V \subset \mathbb{R}^4$ spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Extend this basis to a basis for $\mathbb{R}^4$.

$$\begin{array}{c} v_1 \\ \cancel{v_2} \\ v_3 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{bmatrix} -1 & 2 & -1 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} \text{relation} \\ +1 \\ -1 \\ +1 \\ \\ \\ \\ \\ \end{array}$$

$v_2 = v_1 + v_3$ so v_1, v_2, v_3 dependent

cross out v_2

pick these to make triangular matrix, obviously independent

$\Downarrow$

$$\begin{array}{c} v_1 \\ v_3 \\ e_3 \\ e_4 \end{array} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

determinant $= 1 \neq 0$ so entries are basis for $\mathbb{R}^4$.

Answer:

$\boxed{\{v_1, v_3\} \text{ is a basis for } V}$

$\boxed{\{v_1, v_3, e_3, e_4\} \text{ extends this to a basis for } \mathbb{R}^4}$

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[2] Let A be the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & -2 & 2 \\ 3 & 0 & -3 & 3 \\ 4 & 0 & -4 & 0 \end{bmatrix}.$$

Compute the row space and column space of A.

This zero doesn't fit pattern, so A isn't quite a mult table, so watch out for col, row 4!

rowspace:

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ \cancel{2} & \cancel{0} & \cancel{-2} & \cancel{2} \\ \cancel{3} & \cancel{0} & \cancel{-3} & \cancel{3} \\ 4 & 0 & -4 & 0 \end{bmatrix}$$

cross out dependent rows, left with two.

$\boxed{(1,0,-1,1),\ (4,0,-4,0) \text{ is a basis for row space.}}$

column space:

$$\begin{bmatrix} 1 & \cancel{0} & \cancel{-1} & 1 \\ 2 & \cancel{0} & \cancel{-2} & 2 \\ 3 & \cancel{0} & \cancel{-3} & 3 \\ 4 & \cancel{0} & \cancel{-4} & 0 \end{bmatrix}$$

expect same # basis vectors in answer, because dim row space = dim col space = rank of A.

cross out dependent columns, left with two. ☑

$\boxed{(1,2,3,4),\ (1,2,3,0) \text{ is a basis for column space.}}$

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[3] Let V be the vector space of all twice-differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Let $W \subset V$ be the set of all functions in V which satisfy the differentiable equation $f'' = f$. Show that W is a subspace of V.

Need to show: ① if $f, g \in W$ then $f+g \in W$.
② if $f \in W$, $r \in \mathbb{R}$ then $rf \in W$.

① $f, g \in W \Rightarrow f'' = f$ and $g'' = g$
$\Rightarrow (f+g)'' = f'' + g'' = f+g$
$\Rightarrow f+g \in W$.

② $f \in W$, $r \in \mathbb{R} \Rightarrow f'' = f$
$\Rightarrow (rf)'' = rf'' = rf$
$\Rightarrow rf \in W$.

So W is a subspace.

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[4] Let $\overbrace{\mathbf{v}_1 = (1,1) \text{ and } \mathbf{v}_2 = (-1,1)}^{\text{basis } V}$. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$$L(\mathbf{v}_1) = 2\,\mathbf{v}_1, \quad L(\mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2.$$

Find a matrix that represents L with respect to the usual basis $\underbrace{\mathbf{e}_1 = (1,0),\ \mathbf{e}_2 = (0,1)}_{\text{basis } E}$.

$$B = \underset{V \leftarrow V}{\overset{L}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}}$$

$L(v_1) = 2v_1 = (2,0)_V$

$L(v_2) = v_1 + v_2 = (1,1)_V$

make these columns of B

$$C = \underset{E \leftarrow V}{\overset{\text{identity}}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}}$$

$v_1 = (1,1)_E$

$v_2 = (-1,1)_E$

make these columns of C

$$C^{-1} = \underset{V \leftarrow E}{\overset{\text{identity}}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}} \Big/ 2$$

compute as inverse translation to C (change of coords)

$$A = \underset{E \leftarrow E}{\overset{L}{\begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}}} \Big/ 2 = \underset{E \leftarrow V}{\overset{\text{ident}}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}} \underbrace{\underset{V \leftarrow V}{\overset{L}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}} \underset{V \leftarrow E}{\overset{\text{ident}}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}} \Big/ 2}_{\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} / 2} = \boxed{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}$$

ANSWER

check:

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad ✓ \quad Av_1 = 2v_1$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad ✓ \quad Av_2 = v_1 + v_2$$

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[5] Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that $L(\mathbf{v}) = \mathbf{v}$ for any vector $\mathbf{v}$ in the plane $x + y + z = 0$, and $L(\mathbf{v}) = \mathbf{0}$ for any vector $\mathbf{v}$ in the line $x = y = z$. Find a matrix that represents L with respect to the usual basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$.

choose basis V coords $v_1 = (1,-1,0)$, $v_2 = (0,1,-1)$ in plane $x+y+z=0$

$v_3 = (1,1,1)$ on line $x=y=z$

$$B = \begin{array}{c} L \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ V \leftarrow V \end{array}$$

$L(v_1) = v_1 = (1,0,0)_V$

$L(v_2) = v_2 = (0,1,0)_V$

$L(v_3) = \mathbf{0} = (0,0,0)_V$

$$C = \begin{array}{c} \text{identity} \\ \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\ E \leftarrow V \end{array}$$

$v_1 = (1,-1,0)_E$

$v_2 = (0,1,-1)_E$

$v_3 = (1,1,1)_E$

$$C^{-1} = \begin{array}{c} \text{identity} \\ \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \Big/ 3 \\ V \leftarrow E \end{array}$$

compute as inverse to C

(-over-)

$$A = \begin{array}{c} L \\ \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 3 \\ E \leftarrow E \end{array} = \begin{array}{c} \text{identity} \\ \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\ E \leftarrow V \end{array} \underbrace{\begin{array}{c} L \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ V \leftarrow V \end{array} \begin{array}{c} \text{identity} \\ \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \Big/ 3 \\ V \leftarrow E \end{array}}_{\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} / 3}$$

answer

check:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 3 \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -3 & 0 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 \end{array} \qquad \qquad \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & 0 \end{array}$$

(check all at once)

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$$C^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} / 3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[2 = 2 + 1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[3 = 3 + 2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[3 = \frac{1}{3} 3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

$$\xrightarrow[\substack{1 = 1 - 3 \\ 2 = 2 - 2\,3}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

check:

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} / 3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} / 3 \quad \boxed{\checkmark}$$

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