Exam 1

Linear Algebra, Dave Bayer, October 5, 2000

Name: Solutions

ID: ________ **School:** ________

[1] (6 pts)	[2] (6 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Only work which can be found in this way will be graded.

Please do not use calculators or decimal notation.

[1] Solve the following system of equations:

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 3 & -1 & 0 & 0 & 2 \\ -1 & 3 & -1 & 0 & 1 \\ 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 3 & 2 \end{array}\right] \xRightarrow{(1)\leftrightarrow(2)} \left[\begin{array}{cccc|c} -1 & 3 & -1 & 0 & 1 \\ 3 & -1 & 0 & 0 & 2 \\ 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 3 & 2 \end{array}\right] \cancel{\xRightarrow[\text{then } (2)=(2)-3(1)]{(1)=-(1)} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -1 \\ 0 & & & & \end{array}\right]}$$

better idea!

$$\xRightarrow[\text{then } (2)\leftrightarrow(3) \text{ then } (3)\leftrightarrow(4)]{(1)\leftrightarrow(2)} \left[\begin{array}{cccc|c} -1 & 3 & -1 & 0 & 1 \\ 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 3 & 2 \\ 3 & -1 & 0 & 0 & 2 \end{array}\right] \xRightarrow{(4)=(4)+3(1)} \left[\begin{array}{cccc|c} -1 & 3 & -1 & 0 & 1 \\ 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 3 & 2 \\ 0 & 8 & -3 & 0 & 5 \end{array}\right] \xRightarrow{(4)=(4)+8(2)} \left[\begin{array}{cccc|c} -1 & 3 & -1 & 0 & 1 \\ 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 3 & 2 \\ 0 & 0 & 21 & -8 & 13 \end{array}\right]$$

$$\xrightarrow{(4)=(4)+21(3)} \left[\begin{array}{cccc|c} -1 & 3 & -1 & 0 & 1 \\ 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 55 & 55 \end{array}\right] \xRightarrow{(4)=\frac{1}{55}(4)} \left[\begin{array}{cccc|c} -1 & 3 & -1 & 0 & 1 \\ 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[(3)=(3)-3(4)]{(2)=(2)+(4)} \left[\begin{array}{cccc|c} -1 & 3 & -1 & 0 & 1 \\ 0 & -1 & 3 & 0 & 2 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xRightarrow[(2)=(2)+3(3)]{(1)=(1)-(3)} \left[\begin{array}{cccc|c} -1 & 3 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{(1)=(1)+3(2)} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xRightarrow[(3)=-(3)]{(1)=-(1),\ (2)=-(2)} \left[\begin{array}{cccc|c} 1 & & & & 1 \\ & 1 & & & 1 \\ & & 1 & & 1 \\ & & & 1 & 1 \end{array}\right]$$

so $\boxed{(w,x,y,z) = (1,1,1,1)}$ ⇐ answer

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Check:
$$\begin{bmatrix} 3 & -1 & & \\ -1 & 3 & -1 & \\ & -1 & 3 & -1 \\ & & -1 & 3 \end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\-1\\0\\0\end{bmatrix}1 + \begin{bmatrix}-1\\3\\-1\\0\end{bmatrix}1 + \begin{bmatrix}0\\-1\\3\\-1\end{bmatrix}1 + \begin{bmatrix}0\\0\\-1\\3\end{bmatrix}1 = \begin{bmatrix}2\\1\\1\\2\end{bmatrix}$$

put differently, $A\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ sums the rows of A,

and the row sums are $2, 1, 1, 2$ $\square$

Notice how, in every step of row reduction, the right hand side is always the row sums of the left hand side. In other words $(1,1,1,1)$ is always the solution, to every intermediate step.

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[2] Express the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[①=\frac{1}{2}①]{A} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[②=\frac{1}{2}②]{B} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[③=\frac{1}{2}③]{C} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[①=①-\frac{1}{2}③]{D} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[②=②-\frac{1}{2}③]{E} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[①=①-\frac{1}{2}②]{F} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

so write inverse elementary matrices to each step

$$\boxed{\underset{A^{-1}}{\begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}} \underset{B^{-1}}{\begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}} \underset{C^{-1}}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}} \underset{D^{-1}}{\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}} \underset{F^{-1}}{\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$ ⟸ answer

(this, multiplied ~~by~~ onto final result $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ of row reduction, will <u>reverse</u> row reduction, getting original matrix)

check

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

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Problem: ____

Here is another way a student did it, postpones dividing so answer is simpler:

$$\begin{bmatrix}2&1&1\\0&2&1\\0&0&2\end{bmatrix} \Rightarrow \begin{bmatrix}2&1&1\\0&2&1\\0&0&1\end{bmatrix} \Rightarrow \begin{bmatrix}2&1&1\\0&2&0\\0&0&1\end{bmatrix} \Rightarrow \begin{bmatrix}2&1&0\\0&2&0\\0&0&1\end{bmatrix} \Rightarrow \begin{bmatrix}2&1&0\\0&1&0\\0&0&1\end{bmatrix} \Rightarrow \begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underset{A}{③=\tfrac{1}{2}③} \qquad \underset{B}{②=②-③} \qquad \underset{C}{①=①-③} \qquad \underset{D}{②=\tfrac{1}{2}②} \qquad \underset{E}{①=①-②} \qquad \underset{F}{①=\tfrac{1}{2}①}$$

$$\underbrace{\overset{A^{-1}}{\begin{bmatrix}1&&\\&1&\\&&2\end{bmatrix}}\overset{B^{-1}}{\begin{bmatrix}1&&\\&1&1\\&&1\end{bmatrix}}}_{\begin{bmatrix}1&&\\&1&1\\&&2\end{bmatrix}}\underbrace{\overset{C^{-1}}{\begin{bmatrix}1&&1\\&1&\\&&1\end{bmatrix}}\overset{D^{-1}}{\begin{bmatrix}1&&\\&2&\\&&1\end{bmatrix}}}_{\begin{bmatrix}1&0&1\\0&2&0\\0&0&1\end{bmatrix}}\underbrace{\overset{E^{-1}}{\begin{bmatrix}1&1&0\\&1&\\&&1\end{bmatrix}}\overset{F^{-1}}{\begin{bmatrix}2&&\\&1&\\&&1\end{bmatrix}}}_{\begin{bmatrix}2&1&0\\0&1&0\\0&0&1\end{bmatrix}}$$

another answer

$$\begin{bmatrix}1&0&1\\0&2&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&1&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}2&1&1\\0&2&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&&\\&1&1\\&&2\end{bmatrix}\begin{bmatrix}2&1&1\\0&2&0\\0&0&1\end{bmatrix} = \begin{bmatrix}2&1&1\\0&2&1\\0&0&2\end{bmatrix} \checkmark$$

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[3] Compute the determinant of the following 4×4 matrix:

$$\underbrace{\begin{bmatrix} 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 2 \end{bmatrix}}\ \Big\} n=4$$

What can you say about the determinant of the $n \times n$ matrix with the same pattern?

$$\begin{vmatrix} 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 2 \end{vmatrix} = 2^4 \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} = (-1)^3 2^4 \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

(pull out 4 rows of 2s) (need n-1 swaps to move last row to first)

$$= (-1)^3 2^4 \begin{vmatrix} 3 & 3 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (-1)^3 2^4 \cdot 3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (-1)^3 2^4 \cdot 3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

(add all rows to first) (doesn't affect det) (pull out row of 3's) (subtract first from rest of rows) (doesn't affect det)

$= (-1)^{3+3} 2^4 \cdot 3$ since triangular matrix, determinant is product of diagonal entries

$= 2^4 \cdot 3 = \boxed{48}$ answer

For $n \times n$ case, we'd get $\underbrace{(-1)^{(n-1)+(n-1)}}_{\text{always even power}} \cdot 2^n \cdot (n-1) = \boxed{2^n (n-1)}$

answer $n \times n$

check on two smaller problems, (to catch any sign errors)

$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 = 2^2 \cdot (2-1)$ ☑

$n=2$

$\begin{vmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix} = 8+8=16$

$= 2^3(3-1)$ ☑

must be right!

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[4] Using Cramer's rule, find x satisfying the following system of equations:

$$\begin{bmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$x = \begin{vmatrix} a & 1 & 0 \\ b & \lambda & 1 \\ c & 1 & \lambda \end{vmatrix} \Big/ \begin{vmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{vmatrix}$$

idea: expand first matrix down 1st column, second matrix is just same answer with $a=\lambda$, $b=1$, $c=0$.

$$+a\begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} - b\begin{vmatrix} 1 & 0 \\ 1 & \lambda \end{vmatrix} + c\begin{vmatrix} 1 & 0 \\ \lambda & 1 \end{vmatrix} = \underbrace{a(\lambda^2-1) - b(\lambda) + c}_{\text{1st det}}$$

plug in $a=\lambda$, $b=1$, $c=0$, $\quad \lambda(\lambda^2-1) - \lambda = \lambda^3 - 2\lambda$

so $\boxed{x = \dfrac{a(\lambda^2-1) - b\lambda + c}{\lambda^3 - 2\lambda}}$ $\Leftarrow$ answer

check: Let's make up numbers to test formula:

$\lambda=2$, $(x,y,z) = (1,2,3)$ $\Leftarrow$ do this instead of picking (a,b,c) so we don't have to solve problem in order to check!

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\Leftarrow$ so this (a,b,c) has answer we know, $x=1$. Use it to check.

$$x = \frac{4(2^2-1) - 8\cdot 2 + 8}{2^3 - 2\cdot 2} = \frac{4\cdot 3 - 8}{8-4} = \frac{12-8}{8-4} = \frac{4}{4} = 1 \quad ✓$$

$\lambda=1$, $x,y,z = (2,1,0)$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$x = \frac{3(1^2-1) - 3\cdot 1 + 1}{1^3 - 2\cdot 1} = \frac{-2}{-1} = 2 \quad ✓$$

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do it again, new numbers

[5] Give a formula for the matrix which is inverse to:

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix}^{-1} = \frac{1}{\det} \underbrace{\begin{bmatrix} +\begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & \lambda \end{vmatrix} & +\begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix} \\ 〃 & +\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} & -\begin{vmatrix} \lambda & 1 \\ 0 & 1 \end{vmatrix} \\ 〃 & 〃 & +\begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} \end{bmatrix}}_{\text{matrix is symmetric so don't have to do both halves, and don't have to worry about transposing}} = \frac{1}{\det} \begin{bmatrix} \lambda^2-1 & -\lambda & +1 \\ 〃 & \lambda^2 & -\lambda \\ 〃 & 〃 & \lambda^2-1 \end{bmatrix}$$

$$\det = +\lambda \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 1 & \lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ \lambda & 1 \end{vmatrix} = \lambda(\lambda^2-1) - \lambda = \underbrace{\lambda^3 - 2\lambda}_{\text{hey, didn't we just do this?}}$$

so

$$\boxed{A^{-1} = \frac{1}{\lambda^3 - 2\lambda} \begin{bmatrix} \lambda^2-1 & -\lambda & 1 \\ -\lambda & \lambda^2 & -\lambda \\ 1 & -\lambda & \lambda^2-1 \end{bmatrix}} \Leftarrow \text{answer}$$

(nice pattern!)

check:

$$\begin{bmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^2-1 & -\lambda & 1 \\ -\lambda & \lambda^2 & -\lambda \\ 1 & -\lambda & \lambda^2-1 \end{bmatrix} \Big/ \det = \begin{bmatrix} \lambda(\lambda^2-1)-\lambda & -\lambda^2+\lambda^2 & \lambda-\lambda \\ \lambda^2-1-\lambda^2+1 & -\lambda+\lambda^3-\lambda & 1-\lambda^2+\lambda^2-1 \\ -\lambda+\lambda & \lambda^2-\lambda^2 & -\lambda+\lambda(\lambda^2-1) \end{bmatrix} \Big/ \det$$

$$= \begin{bmatrix} \lambda^3-2\lambda & 0 & 0 \\ 0 & \lambda^3-2\lambda & 0 \\ 0 & 0 & \lambda^3-2\lambda \end{bmatrix} \Big/ (\lambda^3-2\lambda) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} ✓$$

check 2:

$$\frac{1}{\lambda^3-2\lambda} \begin{bmatrix} \lambda^2-1 & -\lambda & 1 \\ -\lambda & \lambda^2 & -\lambda \\ 1 & -\lambda & \lambda^2-1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left\{ \frac{1}{\lambda^3-2\lambda} \begin{bmatrix} a(\lambda^2-1) - b(\lambda) + c \\ \sim\sim \\ \sim\sim \end{bmatrix} = \begin{bmatrix} x \\ 〃 \\ 〃 \end{bmatrix} \right.$$

So this answer agrees w/ our formula in problem 4.

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