Final Exam

Linear Algebra, Dave Bayer, December 21, 2000

Name: ANSWER KEY

ID: ______________________ **School:** ______________________

[1] (5 pts)	[2] (5 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	[6] (6 pts)	[7] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Please do not use calculators or decimal notation.

[1] Compute the determinant of the following 4×4 matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & 0 & 2 & 2 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 4 & 0 \end{bmatrix}$$

What can you say about the determinant of the $n \times n$ matrix with the same pattern?

$$\begin{vmatrix} 0&1&1&1 \\ 2&0&2&2 \\ 3&3&0&3 \\ 4&4&4&0 \end{vmatrix} = 2\cdot3\cdot4 \begin{vmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{vmatrix} = 2\cdot3\cdot4 \begin{vmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 3&3&3&3 \end{vmatrix} = 2\cdot3\cdot4\cdot3 \begin{vmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&1 \end{vmatrix}$$

$(4) = (4)+(1)+(2)+(3)$

$$= 2\cdot3\cdot4\cdot3 \begin{vmatrix} -1&0&0&0 \\ 0&-1&0&0 \\ 0&0&-1&0 \\ 1&1&1&1 \end{vmatrix} = 2\cdot3\cdot4\cdot3\cdot(-1)^3 = 4!\,3(-1)^3 = \boxed{-72}$$

$(1) = (1)-(4)$
$(2) = (2)-(4)$
$(3) = (3)-(4)$

general pattern: $\boxed{n!\,(n-1)(-1)^{n-1}}$

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[2] Let V be the vector space of all polynomials $f(x)$ of degree ≤ 3. Let $W \subset V$ be the set of all polynomials f in V which satisfy $f(0) = f(1) = 0$. Show that W is a subspace of V. Find a basis for W. Extend this basis to a basis for V.

$$V = \{ax^3 + bx^2 + cx + d\} = \{(a,b,c,d)\}$$

$$f(0)=0: \quad d=0$$
$$f(1)=0: \quad a+b+c+d=0$$

2 conditions on 4-space, so W is 2-dimensional.

Basis for W: $\underbrace{(1,-1,0,0)}_{x^3-x^2}$, $\underbrace{(0,1,-1,0)}_{x^2-x}$

Basis for W: $\left\{ \begin{array}{l} x^3-x^2 \\ x^2-x \end{array} \right.$

extend to Basis for V: $\left\{ \begin{array}{l} x \\ 1 \end{array} \right.$

To show W is subspace of V:

① $f(x), g(x) \in W \overset{?}{\Rightarrow} (f+g)(x) \in W$

② $f(x) \in W \overset{?}{\Rightarrow} (rf)(g) \in W$

① $(f+g)(0) = f(0) + g(0) = 0$
$(f+g)(1) = f(1) + g(1) = 0$ ✓

② $(rf)(0) = rf(0) = 0$
$(rf)(1) = rf(1) = 0$ ✓

$\boxed{\text{So } W \text{ is a subspace.}}$

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[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $L(v) = v$ for any vector v in the plane $x + y - z = 0$, and such that $L(1,1,0) = (0,0,0)$. Find the matrix A that represents L in standard coordinates.

$$V \begin{cases} (0,1,1) \mapsto (0,1,1) \\ (1,0,1) \mapsto (1,0,1) \\ (1,1,0) \mapsto (0,0,0) \end{cases} \quad \begin{matrix} \lambda = 1, \text{ in } x+y+z=0 \\ \\ \lambda = 0 \end{matrix}$$

$$\underset{E \leftarrow E}{\overset{L}{\begin{bmatrix} A \end{bmatrix}}} = \underset{E \leftarrow V}{\overset{id}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}} \underbrace{\underset{V \leftarrow V}{\overset{L}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}}} \underset{V \leftarrow E}{\overset{id}{\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}} /2}_{\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}/2}$$

$$\begin{array}{c|ccc} & 0\ 1 & 1 & 0\ 1 \\ \hline 1 & 0 & 1\ \ 1 & 0 \\ & (-1) & (1) & (1) \\ 1 & 1 & 0 & 1 \\ & (1) & (-1) & (1) \\ 0 & 1 & 1\ \ 0 & 1 \\ & (1) & (1) & (-1) \\ 1 & 0 & 1\ \ 1 & 0 \end{array} /2$$

$$\boxed{A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} /2}$$

check:

$$\frac{\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \frac{\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix}}{2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \checkmark$$

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[4] By least squares, find the equation of the form $y = ax + b$ which best fits the data $(x_1, y_1) = (0, -1)$, $(x_2, y_2) = (1, 1)$, $(x_2, y_2) = (2, 1)$, $(x_3, y_3) = (3, -1)$.

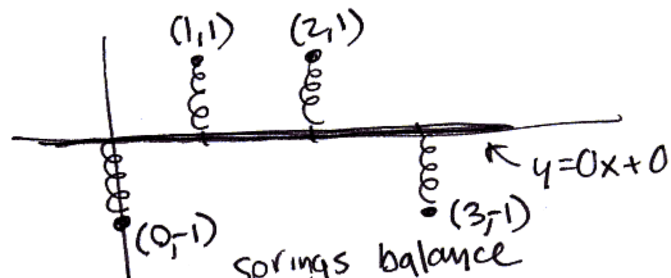
$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has solution $(a,b) = (0,0)$

$$\boxed{y = 0x + 0}$$

check:



springs balance

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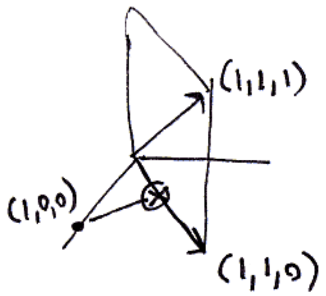
[5] Find (s,t) so $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$ is as close as possible to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.



$(s,t) = (\frac{1}{2}, 0)$ maps to $(\frac{1}{2}, \frac{1}{2}, 0)$ $\otimes$
closest to $(1,0,0)$ by visual inspection.

check by full computation:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\boxed{(s,t) = (\frac{1}{2}, 0)}$ is clearly the solution.

[6] Find an orthogonal basis for the subspace $w + x + y + z = 0$ of $\mathbb{R}^4$.

Codimension 1 = dimension 3.

"Seed" basis: rows $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{matrix} = u_1 \\ = u_2 \\ = u_3 \end{matrix}$

Now Gram-Schmidt

$$\left.\begin{aligned} v_1 &= (1,-1,0,0) \\ v_2 &= (0,0,1,-1) \end{aligned}\right\} \text{in this order, already } \perp$$

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (0,1,-1,0) - \frac{(-1)}{2}(1,-1,0,0) - \frac{(-1)}{2}(0,0,1,-1)$$

$$= \left(\tfrac{1}{2}, \tfrac{+1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}\right) \text{ scale to } (1,1,-1,-1)$$

rows of: $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$

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[7] Let A be the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

Find a basis of eigenvectors and eigenvalues for A. Find the matrix exponential e^A.

$$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$$

$$\lambda^2 - 25 = 0$$

$$(\lambda-5)(\lambda+5) = 0 \quad \boxed{\lambda_i = \pm 5} \text{ are eigenvalues.}$$

$$\lambda = 5: \begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \quad \boxed{v_1 = (2,1)}$$

$$\lambda = -5: \begin{bmatrix} 3+5 & 4 \\ 4 & -3+5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \quad \boxed{v_2 = (1,-2)}$$

id

$$A = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}_{E \leftarrow V} \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}}_{V \leftarrow V} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}_{V \leftarrow E} /5$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} ✓$$

$$e^A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} e^5 & 0 \\ 0 & e^{-5} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}_{} /5$$

$$\begin{bmatrix} 2e^5 & e^5 \\ e^{-5} & -2e^{-5} \end{bmatrix} /5$$

$$\boxed{e^A = \begin{bmatrix} 4e^5 + e^{-5} & 2e^5 - 2e^{-5} \\ 2e^5 - 2e^{-5} & e^5 + 4e^{-5} \end{bmatrix} /5}$$

check: e^{At} change e^5 to e^{5t}, e^{-5} to e^{-5t}

$\Downarrow$

$t=0$ get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$\frac{d}{dt}$ then $t=0$ get $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ ✓

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