Name and UNI: ________________________________

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Instructions:

- There are 4 questions on this exam.
- Part (e) of question 2 is bonus. You DO NOT have to solve it to get full credit.
- Please write your NAME and UNI on top of EVERY page.
- SHOW YOUR WORK in every question.
- Please write neatly, and put your final answer in a box.
- No calculators, cell phones, books, notebooks, notes or cheat sheets are allowed.
- Useful identities:
  \[ \sin(2\theta) = 2\sin(\theta)\cos(\theta) \]
  \[ \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \]
1. (a) (3 points) Let \( p \in \mathbb{R} \) denote a constant. Set up an integral that calculates the volume of the solid obtained by rotating the area enclosed by the curves

\[
y = \frac{1}{x^p} \quad , \quad y = 0 \quad , \quad x = 0 \quad , \quad x = 1
\]

around the \( y \)-axis.

(b) (5 points) Determine for which values of \( p \) the volume is finite (i.e. For which \( p \) does the integral in part (a) converge?).

(c) (2 points) For those values of \( p \) for which the integral converges calculate the integral.

Solution:

(a) Since the equations defining the region are given in terms of \( y = f(x) \) and we are rotating around the \( y \)-axis it is easier to use the cylindrical shells. The radius of a typical shell is \( x \), the height is \( \frac{1}{x^p} \), and we are integrating from \( x = 0 \) to \( x = 1 \). Then the volume is given by

\[
2\pi \int_0^1 \frac{dx}{x^{p-1}}
\]

(b) First consider the indefinite integral

\[
\int \frac{dx}{x^{p-1}}.
\]

This integral will behave differently depending on \( p - 1 = 1 \) or not.

- \( p \neq 2 \). In this case,

\[
\int \frac{dx}{x^{p-1}} = \frac{1}{2-p} \frac{1}{x^{p-2}}.
\]

Therefore we have,

\[
\int_0^1 \frac{dx}{x^{p-1}} = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{dx}{x^{p-1}} = \frac{1}{2-p} \lim_{\epsilon \to 0^+} (1 - \frac{1}{x^{p-2}}).
\]

The last limit is 0 if \( p < 2 \) and is \( \infty \) if \( p > 2 \). Therefore we conclude that the integral converges when \( p < 2 \).

- \( p = 2 \). In this case,

\[
\int \frac{dx}{x^{p-1}} = \ln |x|.
\]

Therefore we have,

\[
\int_0^1 \frac{dx}{x} = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{dx}{x} = \lim_{\epsilon \to 0^+} (-\ln \epsilon) = \infty.
\]

So the integral is divergent.

In conclusion the volume is finite if and only if \( p < 2 \).

(c) By the calculation above, when \( p < 2 \) the volume is \( \frac{2\pi}{p-2} \).
2. Consider the curve, \( C \), given by the parametric equation
\[
C = \{ x = t^3 - 3t, \ y = 3t^2 - 9 \mid -3 \leq t \leq 3 \}.
\]
(a) (2 points) Find \( x \) and \( y \)-intercepts of this curve.
(b) (4 points) Find \( \frac{dy}{dx} \) and determine where the curve is increasing and where it is decreasing.
(c) (4 points) Find \( \frac{d^2y}{dx^2} \) and determine where the curve is concave up and where it is concave down.
(d) (2 points) Sketch the curve.
(e) (Bonus (5) points) Find the area that is inside the closed loop of the curve.

Solution:

(a) \( x \)-intercepts.
\[
y = 0 \iff 3t^2 - 9 = 0
\]
\[
\iff (t^2 - 3) = 0
\]
\[
\iff t = \pm \sqrt{3}.
\]
Therefore the \( x \)-intercepts are given by \((0, 0)\).

\( y \)-intercepts.
\[
x = 0 \iff t^3 - 3t = 0
\]
\[
\iff t(t^2 - 3) = 0
\]
\[
\iff t = 0, \pm \sqrt{3}.
\]
Therefore the \( y \)-intercepts are given by \((0, 0)\) and \(0, -9)\).

(b)
\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{3(t^2 - 1)} = \frac{2t}{(t+1)(t-1)}
\]
The critical values are \( t = 0, \pm 1 \). The function is decreasing for \( t \) the intervals \((-\infty, -1) \cup (0, 1)\) and increasing in \((-1, 0) \cup (1, \infty)\).

(c)
\[
\frac{d^2y}{dx^2} = \frac{\frac{d(dy/dt)}{dt}}{\frac{dx}{dt}} = \frac{-2(t^2+1)}{3(t^2-1)^2}
\]
Since the numerator is always negative the sign of this is determined by the sign of the denominator. This implies that the function is concave down on \( t \in (-\infty, -1) \cup (1, \infty) \) and concave up on \( t \in (-1, 1) \).

(d)
(e) To find the area of the closed loop we first realize that it is twice the area of the right side of the loop which is traced when $t \in (-\sqrt{3}, 0)$. In order to calculate the area of the right half we integrate in the $y$-direction.

\[
\text{Area} = 2 \int_{-9}^{0} x \, dy \\
= 2 \int_{0}^{-\sqrt{3}} (t^3 - 3t) \, dt \\
= 12 \int_{0}^{-\sqrt{3}} (t^4 - 3t^2) \, dt \\
= \frac{72\sqrt{3}}{5}
\]
3. Consider the differential equation

\[ xy \frac{dy}{dx} - y^2 = \sqrt{x^2 + y^2}, \quad x > 0. \]

(a) (1 point) Determine if this equation is linear, separable, or neither.

(b) (4 points) Use the change of variables

\[ y = xu \]

to rewrite the equation in terms of \( x \) and \( u \).

\( \text{(Hint: Don’t forget that } u \text{ is a function of } x \text{. When rewriting the derivative } \frac{dy}{dx} \text{ in terms of the derivative } \frac{du}{dx} \text{ (and possibly other things depending only on } u \text{ and } x \text{) you will need to use the product rule.)} \)

(c) (4 points) Find the general solution to this equation in terms of \( u \).

(d) (1 point) Substitute back \( u = \frac{y}{x} \) to get the general solution of the original equation.

**Solution:**

(a) The equation is neither linear nor separable.

(b) Note that,

\[ y = ux \Rightarrow y' = u + xu'. \]

Substituting this back into the equation we get,

\[ x(ux)(u + xu') - (ux)^2 = \sqrt{x^2 + (ux)^2} \]

\[ \Leftrightarrow x^3 uu' = |x| \sqrt{1 + u^2} \]

\[ \Leftrightarrow x^2 uu' = \sqrt{1 + u^2}. \]

Where, in the last line we used that \( x > 0 \) so \(|x| = x\).

(c) This equation is separable.

\[ x^2 uu' = \sqrt{1 + u^2} \]

\[ \Leftrightarrow \int \frac{udu}{\sqrt{1 + u^2}} = \int \frac{dx}{x} \]

\[ \Leftrightarrow \sqrt{1 + u^2} = \frac{1}{x} + C. \]

(d) \[
\sqrt{1 + \left(\frac{y}{x}\right)^2} = \frac{1}{x} + C
\]
4. Consider the curve given by

\[(x^2 + y^2)^{\frac{3}{2}} = x^2 - y^2.\]

Whose graph is

(a) (3 points) Taking \(r > 0\) and \(0 \leq \theta < 2\pi\) write this equation in polar coordinates.

(Hint: You may find it useful to use the formula given on the first page for \(\cos^2(\theta) - \sin^2(\theta)\).)

(b) (3 points) Write a parametric equation for this curve using the polar equation you have found in part (a).

(c) (4 points) Using the parametric equation you have found in part (b) calculate the area of one of the leaves of the curve.

Solution:

(a) Recall \(x = r \cos(\theta)\) and \(y = r \sin(\theta)\). Substituting these we get

\[r^3 = r^2(\cos^2(\theta) - \sin^2(\theta))\]

\(\iff r = \cos(2\theta)\)

(b) The parametric equation is,

\[x = \cos(2\theta) \cos(\theta)\]
\[y = \cos(2\theta) \sin(\theta).\]

(c) We calculate the leaf on the right. By symmetry we can calculate the upper part of the leaf and multiply it by 2. In order to calculate the upper part of the area we integrate \(y\) from 0 to 1.

\[\text{Area} = 2 \int_0^1 y \, dx\]

\[= 2 \int_{\pi/4}^0 \cos(2\theta) \sin(\theta)(-2 \sin(2\theta) \cos(\theta) - \cos(2\theta) \sin(\theta)) \, d\theta\]

\[= -2 \int_{\pi/4}^0 \cos(2\theta) \sin(2\theta) 2 \sin(\theta) \cos(\theta) \, d\theta - 2 \int_{\pi/4}^0 \cos^2(2\theta) \sin^2(\theta) \, d\theta\]

\[= -2 \int_{\pi/4}^0 \cos(2\theta) \sin^2(2\theta) \, d\theta - 2 \int_{\pi/4}^0 \cos^2(2\theta) (\frac{1-\cos(2\theta)}{2}) \, d\theta\]

\[= -2 \int_{\pi/4}^0 \cos(2\theta) \sin^2(2\theta) \, d\theta - \int_{\pi/4}^0 \cos^2(2\theta) \, d\theta + \int_{\pi/4}^0 \cos^3(2\theta) \, d\theta\]

\[= -2 \int_{\pi/4}^0 \cos(2\theta) \sin^2(2\theta) \, d\theta - \frac{1}{2} \int_{\pi/4}^0 (\cos(4\theta) + 1) \, d\theta + \int_{\pi/4}^0 \cos(2\theta)(1 - \sin^2(2\theta)) \, d\theta\]

\[= \frac{-\sin^3(2\theta)}{3} \bigg|_{\pi/4}^0 - \frac{\sin(4\theta)}{8} - \frac{\theta}{2} + \frac{\sin(2\theta)}{2} - \frac{\sin^3(2\theta)}{6} \bigg|_{\pi/4}^0\]

\[= \frac{\pi}{8}\]