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1 HW1, Due on Feb 5th

Some of the questions below need Newton polygon. Let $K$ be a non-archimedean local field (with valuation normalized such that a uniformizer has valuation equal to one). Let $f(X) = a_nX^n + \ldots + a_1X + a_0 \in K[X]$ be a polynomial of degree $n$. The Newton polygon is defined to be the lower convex hull on the $(x,y)$-plane of the set of points $(i, v(a_i)), i = 0, \ldots, n$. Let $\lambda_1, \ldots, \lambda_r$ be the slopes of the line segments of the Newton polygon and denote the lengths of the corresponding line segments projected onto the x-axis by $m_1, \ldots, m_r$ (so
that $m_1 + \ldots + m_r = n$). Then there are precisely $m_i$ number of roots (in $K$) of $f(x)$ with valuation equal to $-\lambda_i$ for every $i = 1, 2, \ldots, r$. You may use this result freely.

1. Show that $\mathbb{Q}_p$ has no nontrivial field automorphism (i.e., there is no $\sigma \neq id : \mathbb{Q}_p \to \mathbb{Q}_p$ as an isomorphism of fields, where we don’t assume that $\sigma$ is continuous).

2. Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ its completion with respect to the unique extension of absolute value of $\mathbb{Q}_p$.

   (a) Consider the series $\sum_{n=1}^{\infty} \zeta_n p^n$ where $\zeta_n' = \zeta_n$ is any primitive $n'$-th root of unity if $(n, p) = 1$ and otherwise $n' = 1$. Show that the series converges in $\mathbb{C}_p$ but the limit is not inside $\overline{\mathbb{Q}}_p$. In particular, $\overline{\mathbb{Q}}_p$ is not complete.

   (b) Show that $\mathbb{C}_p$ is algebraic closed. (Hint: Krasner lemma.)

3. Which of the following polynomials are irreducible over $\mathbb{Q}_5$?

   \[ X^3 + 5X + 125, \quad X^3 + 5X + 25, \quad X^3 + 5X^2 + 25, \]

   \[ X^4 + 5X^2 + 25, \quad X^4 + 15X^2 + 25. \]

4. Consider the polynomial $f(X) = X^p - X - 1/p \in \mathbb{Q}_p[X]$.

   (a) Show that $f(X)$ is irreducible over $\mathbb{Q}_p$.

   (b) Let $\alpha$ be a root of $f$ and denote $K = \mathbb{Q}_p(\alpha)$ with integer ring $\mathcal{O}_K$ and maximal ideal $m_K$. Show that for $i = 0, 1, \ldots, p - 1$, there exists a unique element $\beta_i \in \mathcal{O}_K$ such that $\alpha + \beta_i$ is a root of $f(X)$ and $\beta_i \equiv i \pmod{m_K}$. In particular, $K/\mathbb{Q}_p$ is Galois.

2. **HW2, due on Feb 12th**

Let $K$ be a non-archimedean local field, i.e., a finite extension of $\mathbb{Q}_p$ (the mixed characteristic case), or the field of Laurent series $\mathbb{F}_q((\pi))$. For a ring $R$ (with 1), recall that a (commutative one-dimensional) formal group $F$ over $R$ is a power series $F(X, Y)$ such that
1. $F(X,Y) = X + Y + \text{terms of degree at least 2}.$

2. $F(X,F(Y,Z)) = F(F(X,Y),Z).$

3. There exists a unique $i_F \in X \cdot R[[X]]$ such that $F(X,i_F(X)) = 0.$

4. $F(X,Y) = F(Y,X).$

Let $R$ be an $\mathcal{O}_K$-algebra, with the structure ring homomorphism $i : \mathcal{O}_K \rightarrow R$. A formal group $F$ over $R$ is always assumed to be commutative, and one-dimensional. A formal $\mathcal{O}_K$-module over an $\mathcal{O}_K$-algebra $R$ is the data $(F,i)$ where $F$ is a formal group $F$ with a ring homomorphism $i : \mathcal{O}_K \rightarrow \text{End}_R(F)$ such that $d\iota : \mathcal{O}_K \rightarrow \text{Lie}_R(F) \simeq R$ coincides with the structure homomorphism $i : \mathcal{O}_K \rightarrow R$.

1. “Local Hermite-Minkowski theorem”.
   (a) For a fixed $N$, show that there are only finitely many unramified field extensions of $\mathbb{Q}_p$ of degree at most $N$.
   (b) For a fixed $N$, there are only finitely many field extensions of $\mathbb{Q}_p$ of degree at most $N$. (Hint: Eisenstein polynomial and Krasner lemma.)

2. Let $F(X,Y)$ be a formal group. Show that $F(X,0) = X$ and $F(0,Y) = Y$ (namely, there is no terms like $X^m$, $m > 1$). Deduce that the condition on the existence of $i_F(X)$ in the definition of formal groups follows from the others.

3. Let $R$ be $\mathcal{O}_K$ or a field of arbitrary characteristic $p = 0$ or a prime number. Let $\mathbb{G}_a$ be the formal additive group over $R$. What is the endomorphism algebra of $\mathbb{G}_a$?

4. Let $R$ be a commutative $\mathbb{Q}$-algebra and $F$ be a formal group defined over $R$. Show that there is a unique isomorphism, called the logarithm of $F$

$$\log_F : F \rightarrow \mathbb{G}_a$$

such that $\log_F(X) \equiv X \pmod{\text{deg} \geq 2}$. In particular, any formal group over a $\mathbb{Q}$-algebra is isomorphic to the formal additive group. If
$F = \mathbb{G}_m$ is the formal multiplicative group, then
\[
\log_F(X) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{X^i}{i} = \log(1 + X).
\]

5. Now $R = \mathcal{O}_K$. Set
\[
f(X) = X + \pi^{-1}Xq + \pi^{-2}Xq^2 + \ldots = \sum_{i=0}^{\infty} \pi^{-i}Xq^i.
\]
Show that $F(X,Y) := f^{-1}(f(X) + f(Y))$ lies in $R[[X,Y]]$ and defines a formal $\mathcal{O}_K$-module with $\mathcal{O}_K$-action given by
\[
[a]_F(x) = f^{-1}(a \cdot f(X)), \quad a \in \mathcal{O}_K,
\]
with $\log_{F_K} = f$, where $F_K$ denotes the base change of $F$ from $R = \mathcal{O}_K$ to $F$.

6. Let $R = \mathcal{O}_K$, and $f \in \mathcal{F}_\pi$ a Lubin-Tate power series. Let $(F_f, \iota)$ be the Lubin-Tate formal $\mathcal{O}_K$-module over $R$, with $\iota : \mathcal{O}_K \to \text{End}_R(F_f)$.

(a) Show that $\text{End}_R(F_f, \iota) \simeq \mathcal{O}_K$.

(b) Let $K/L$ be an unramified quadratic extension. Consider the formal $\mathcal{O}_L$ module $(F_f, \iota|_{\mathcal{O}_L})$. Base change to $R = \overline{k}$ (viewed as an $\mathcal{O}_L$-algebra). Compute the endomorphism of $\text{End}_{\overline{k}}(F_f, \iota|_{\mathcal{O}_L})$.

3 HW3, due on Feb 19th

The first four questions are concerned with the Lubin-Tate formal module over a local field of positive characteristic. Let $K$ be the field of Laurent series $\mathbb{F}_q((\pi))$ and fix a separable closure of $K$. Let $f$ be the Lubin-Tate power series
\[
f(T) = \pi T + T^q.
\]
The Lubin-Tate formal $\mathcal{O}_K$-module $(F, \iota)$ over $\mathcal{O}_K$ is easy to describe as the formal additive group $F(X,Y) = X + Y$ and $\iota : \mathcal{O}_K \to \text{End}(F)$ characterized by
\[
[a]_f(X) = aX, a \in \mathbb{F}_q
\]
and
\[
[\pi]_f(X) = \pi X + X^q.
\]
1. Let $t_1$ be a nonzero root of $f(T) = 0$, and $t_{n+1}$ a root of $f(T) - t_n = 0$ for $n > 2$. Let $K_{\pi,n} = K(t_n)$ and $K_{\pi} = \bigcup_n K_{\pi,n}$ the Lubin-Tate extension. Let $L = \widehat{K}_{\pi}$ be the completion of $K_{\pi}$. Show that the sequence
\[ t_1^q, t_2^q, \ldots \]
is a Cauchy sequence, hence converges in $L$ (but not in $K_{\pi}$).

2. Denote by $t$ the limit of the sequence above. Show that $t^{q^{-n}}$ exists in $L$, for all $n > 1$.

3. Lubin–Tate theory asserts that $\phi_K : \mathcal{O}_K^\times \cong \text{Gal}(K_{\pi}/K)$ where an element $a \in \mathcal{O}_K^\times$ acts on $K_{\pi}$ by $\phi_K(a)(t_n) = [a]_f(t_n)\textsuperscript{1}$. This action extends continuously to $L$. Show that, if $a = \sum_{i=0}^{\infty} a_i\pi^i \in \mathcal{O}_K^\times$, $a_i \in \mathbb{F}_q$, then
\[
\phi_K(a)(t) = \sum_{i=0}^{\infty} a_i t^{q^i}.
\]

4. Let $\mathbb{F}_q[t^{q^{-\infty}}]$ denote the ring $\mathbb{F}_q[t, t^{q^{-1}}, \ldots, t^{q^{-n}}, \ldots]$. Let $\mathbb{F}_q[t^{q^{-\infty}}]$ denote the completion of the ring $\mathbb{F}_q[t^{q^{-\infty}}]$ with respect to the $t$-adic topology:
\[
\mathbb{F}_q[t^{q^{-\infty}}] := \lim_n \mathbb{F}_q[t^{q^{-n}}]/(t^n).
\]
Concretely, $\mathbb{F}_q[t^{q^{-\infty}}]$ consists of $\sum_{i \in I} a_i t^i$ where $I$ is a subset of $\mathbb{Z}[1/q]_{\geq 0}$, and for every $C > 0$ there are only finitely many nonzero $a_i$ such that $i < N$.

Show that $\mathcal{O}_L/(t) = \mathbb{F}_q[t^{q^{-\infty}}]/(t)$, and $\mathcal{O}_L = \mathbb{F}_q[t^{q^{-\infty}}]$. Deduce from this that $L$ is a perfect field (i.e., the Frobenius is an isomorphism). (This is an example of a perfectoid field in characteristic $p$).

The following two questions are on group cohomology. They are in the texts of Milne, but you should try them yourselves before looking there (if you have-not seen them before).

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\textsuperscript{1}We invert the usual local Artin map such that the uniformizers are sent to the geometric Frobenius, i.e., the automorphism of $K^{ur} = \overline{\mathbb{F}}_q((\pi))$ given by the geometric Frobenius on $\overline{\mathbb{F}}_q$ sending $\alpha \in \overline{\mathbb{F}}_q \mapsto \alpha^{1/q}$. 

5
1. Let $G$ be a cyclic group with a generator $\sigma$. Denote by $Nm_G$ the norm map: $M \to M$ sending $m$ to $\sum_{g \in M} gm$. Show that $H^1(G, M) \simeq \ker(Nm_G : M \to M)/(\sigma - 1)M$.

2. Hilbert 90. Let $L/K$ be a finite Galois extension with Galois group $G$ so that $G$ acts on $L^\times$. Show that $H^1(G, L^\times) = 0$.

4 HW4, due on Feb 26th

1. Let $G$ be a finite group and $G_p$ any $p$-Sylow subgroup. Consider the restriction maps $\text{Res}_p : H^r(G, M) \to H^r(G_p, M)$. For $x \in H^r(G, M)$, show that $x = 0$ if and only if $\text{Res}_p(x) = 0$ for all primes $p$.

2. Let $H$ be a subgroup of $G$ and $M$ an $H$-module. Let $c\text{Ind}_H^G M$ be the “compact”-induction, i.e. $c\text{Ind}_H^G M = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$. Show Shapiro’s Lemma for homology:
   $$H_i(G, c\text{Ind}_H^G M) = H_i(H, M)$$
   for all $i \geq 0$.

3. Let $G = \text{PSL}(2, \mathbb{Z})$. Recall that $G$ has a free subgroup $H \cong \mathbb{F}_2$ of index 6, generated by
   $$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$
   You do not need to show this–however, use this to show that for every $G$-module $A$, and for every $x \in H^i(G, A)$, $i \geq 2$, we have $6x = 0$. (Hint: the augmentation ideal for a free group is a free module.)

4. (Cohomology of $\mathfrak{g} = \widehat{\mathbb{Z}}$) Let $\mathfrak{g} = \widehat{\mathbb{Z}}$, the profinite group (the inverse limit $\lim_n \mathbb{Z}/n\mathbb{Z}$). Let $M$ be a $\mathfrak{g}$-module (with continuous action, i.e., the stabilizer of every $m \in M$ is an open subgroup of $\mathfrak{g}$, equivalently $M = \cup_H M^H$ is the union of invariants of open subgroups of $\mathfrak{g}$). Recall from Milne notes that the cohomology of a pro-finite group $G$ acting continuously on $M$ can be computed as the inductive limit, via the inflations, over all open normal subgroups $H$:
   $$H^i(G, M) = \lim_H H^i(G/H, M^H).$$
(a) Denote by $\sigma$ the topological generator 1 in $g$. Let $M'$ be the set of elements annihilated by $1 + \sigma + \sigma^2 + \ldots + \sigma^n$ for some $n \in \mathbb{N}$. Show that

$$H^1(g, M) = \frac{M'}{(\sigma - 1)M}.$$ 

In particular if $M$ is torsion, we have $H^1(g, M) = \frac{M}{(\sigma - 1)M}$.

(b) If $M$ is divisible (multiplication by $n$ is surjective for all $n \in \mathbb{N}$) or finite torsion, then

$$H^2(g, M) = 0.$$ 

(c) Show that $H^i(g, \mathbb{Q}) = 0$, for all $i \geq 0$. Here $\mathbb{Q}$ has the trivial $g$ action.

Below are some questions that are not required to hand in.

1. (Transfer/Verlagerung) Suppose that $H \subset G$ is a subgroup, not necessarily of finite index, and let $A$ be a $G$ module. Recall that the inclusion $i : H \hookrightarrow G$ induces natural maps $i^* : H^i(G, A) \to H^i(H, A)$ and $i_* : H_i(H, A) \to H_i(G, A)$. We often denote the map $i^*$ by Res (restriction), since it corresponds to restriction of cocycles. Now suppose that $H \subset G$ is a finite index subgroup, with $[G : H] = n$. As shown in class, the map

$$N_{G/H} : H^0(H, A) \to H^0(G, A)$$

$$a \mapsto \sum_{s \in G/H} sa$$

is well-defined, and extends cohomologically to a map, which we denote by $CoRes$ (corestriction)

$$CoRes : H^q(H, A) \to H^q(G, A)$$

In addition, it was shown in class that

$$CoRes \circ Res = n$$

There is a corresponding construction in homology: set

$$N'_{G/H} : H_0(G, A) \to H_0(H, A)$$

$$a \mapsto \sum_{s \in G/H} s^{-1}a$$
Check that this is well-defined and show it extends to all homology groups $H_q$. What is the analogue of $\text{CoRes} \circ \text{Res} = n$ in this context? Prove the corresponding identity. Finally, note that, taking $A = \mathbb{Z}$ and $q = 1$, this construction gives a map

$$V : G^{ab} \to H^{ab}$$

What is this map explicitly?

2. Let $G$ be a finite cyclic group. Describe the 2-periodicity of Tate cohomology for cyclic groups in terms of cup product with an element of $H^2(G, \mathbb{Z}) \simeq H^1(G, \mathbb{Q}/\mathbb{Z})$.

5 \ HW5, due on Mar 4th

Recall that the Brauer group of a field $K$ is defined as

$$\text{Br}_K := H^2(\text{Gal}(\overline{K}/K), \overline{K}^\times) = \lim_{L/K, \text{finite Galois}} H^2(L/K, L^\times)$$

where $\overline{K}$ is a separable closure of $K$. We say that an element $\alpha \in B_K$ is split in an extension $L$ of $K$ if it is in the kernel of $\text{Res} : \text{Br}_K \to \text{Br}_L$.

1. Let $k$ be a field of characteristic $p$, and let $K$ be $k^{1/p}$. Show that an element $\alpha \in \text{Br}_k$ is split by $K$ if and only if $p\alpha = 0$. Deduce that the $p$-primary component of $\text{Br}_k$ is trivial if $k$ is perfect (i.e., the $p$-th power map is bijective).

2. Let $K/k$ be a field extension of degree $n$. Let $\alpha \in \text{Br}_k$ be split by $K$. Show that $n\alpha = 0$. (Hint: treat the separable and purely inseparable cases separately.)

The following questions aim to prove that the maps $\phi_{L/K}$ can be glued to form a map $\phi_K : K^\times \to \text{Gal}(K^{ab}/K)$.

1. Let $G$ be a finite group. We know that there is a natural isomorphism

$$H^{-2}(G, \mathbb{Z}) \simeq G^{ab}.$$
Identify $H^{-1}(G, \mathbb{Q}/\mathbb{Z})$ with $\frac{1}{|G|} \mathbb{Z}/\mathbb{Z}$, viewed as a subgroup of $\mathbb{Q}/\mathbb{Z}$. Consider the cup product
\[ \cup : \hat{H}^{-2}(G, \mathbb{Z}) \otimes H^1(G, \mathbb{Q}/\mathbb{Z}) \to \hat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) \simeq \frac{1}{|G|} \mathbb{Z}/\mathbb{Z}. \]

Show that for $s \in G^\text{ab}$ resp. $\chi \in \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$, viewed as elements in $\hat{H}^{-2}(G, \mathbb{Z})$ resp. $H^1(G, \mathbb{Q}/\mathbb{Z})$, we have
\[ s \cup \chi = \chi(s). \]

2. Let $L/K$ be a Galois extension of non-archimedean local fields. Consider the (norm residue) isomorphism
\[ \phi_{L/K} : \hat{H}^0(G, L^\times) = K^\times/\text{N}L^\times \to \hat{H}^{-2}(G, \mathbb{Z}) = G^\text{ab} \]
defined by inverting the cup product with the fundamental class $u_{L/K} \in H^2(G, L^\times) = \text{Br}_{L/K}$. Let $\alpha \in K^\times$ and $\overline{\alpha}$ its class in $K^\times/\text{N}L^\times = \hat{H}^0(G, L^\times)$. Now take a character $\chi \in H^1(G, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ and denote by $\delta \chi \in H^2(G, \mathbb{Z})$ its image under the connecting map
\[ \delta : H^1(G, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z}) \]
arising from the short exact sequence
\[ 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0. \]

Show that
\[ \chi(\phi_{L/K}(\alpha)) = \text{inv}_K(\overline{\alpha} \cup \delta \chi) \in \mathbb{Q}/\mathbb{Z}. \]

3. Deduce from the last question that if we have a tower of abelian extension non-archimedean local fields
\[ K \subset L' \subset L, \]
then for $\alpha \in K^\times$, we have
\[ \phi_{L/K}(\alpha)|_{L'} = \phi_{L'/K}(\alpha). \]

Here we extend $\phi_{L/K}$ (resp. $\phi_{L'/K}$) as maps from $K^\times$ to $\text{Gal}(L/K)$ (resp. $\text{Gal}(L'/K)$).
6 HW6, due on Mar 23th

Below $K$ is a finite extension of $\mathbb{Q}_p$, $\Gamma_K = \text{Gal} (\overline{K}/K)$, $M$ is a finite $\Gamma_K$-module, $H^i(M) = H^i(K, M) = H^i(\Gamma_K, M)$. The local Tate duality asserts that the cup product induces a perfect pairing between two finite abelian groups

$$\cup : H^i(K, M) \times H^i(K, M^*) \to \mathbb{Q}/\mathbb{Z},$$

where $M^* = \text{Hom}(M, \mu_\infty)$, $\mu_\infty = \bigcup_n \mu_n$. The Euler–Poincaré characteristic is defined for any finite $\text{Gal}(\overline{K}/K)$-module $M$:

$$\chi(M) = \frac{h^0(M)h^2(M)}{h^1(M)}, \quad h^i = \#H^i(M).$$

Then we have a formula

$$\chi(M) = \frac{1}{\#O_K/nO_K}, \quad n = \#M.$$

1. Applying Tate’s local duality theorem to the module $\mathbb{Z}/n\mathbb{Z}$ we have a pairing

$$\text{Hom}(G_K, \mathbb{Z}/n\mathbb{Z}) \times K^\times/(K^\times)^n \to \mathbb{Q}/\mathbb{Z}.$$ 

When $K$ contains a primitive $n$-th root of unity $\zeta$, this pairing gives rise to the Hilbert symbol (How?)

$$(, ) : K^\times/(K^\times)^n \times K^\times/(K^\times)^n \to \mathbb{Q}/\mathbb{Z}$$

Confirm that this agrees with the more commonly given definition of the Hilbert symbol, which we recall below. [The usual definition of the Hilbert symbol is as follows. Given $a, b \in K$, we identify $a \in K^\times/(K^\times)^n$ with the character $\chi_a \in \text{Hom}(G_K, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) = H^1(K, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ defined by fixing an $n$-th root $\sqrt[n]{a}$ of $a$, then defining for $\sigma \in G_K$, $\chi_a(\sigma)$ by

$$\sigma(\sqrt[n]{a}) = \zeta^{(n\chi_a(\sigma))}.$$ 

Then we set, for $a, b \in K^\times/(K^\times)^n$

$$(a, b) = b \cup \delta \chi_a \in H^2(K, \overline{K}^\times) = \mathbb{Q}/\mathbb{Z}$$

where $\delta$ is the connecting map in cohomology coming from the exact sequence of trivial $G_K$ modules $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$]
2. Let $G$ be a finite group, and suppose $V$ and $V'$ are finitely generated $\mathbb{Z}_p[G]$-modules such that $V \otimes \mathbb{Q}_p = V' \otimes \mathbb{Q}_p$. Show that one has the following identity in the Grothendieck group $K(G)$:

$$[V/pV] - [pV] = [V'/pV'] - [pV']$$

Here $pV$ denotes the $p$-torsion in $V$. [Hint: Reduce to the case where $V \supset V' \supset pV$, and use the exact sequence $0 \to pV' \to pV \to V/V' \to V'/pV' \to V/pV \to V/V' \to 0$.]

3. How many elements does $K^\times / (K^\times)^2$ have? How many different quadratic extension does $K$ have? In general, how many elements does $K^\times / (K^\times)^\ell$ have for a prime $\ell$?

4. Show that the group $G = \tilde{\mathbb{Z}}$ has cohomological dimension one, i.e., for all finite $G$-module $M$, we have $H^i(G, M) = 0$ for $i > 1$, and there exists $M$ such that $H^1(G, M) \neq 0$.

5. (Local Tate duality for archimedean local fields) Let $K$ be an archimedean local field, and $G = Gal(\bar{K}/K)$. For a finite $G$-module $M$, write $M^* = \text{Hom}(M, \bar{K}^\times)$ as a $G$-module. Show that when $K = \mathbb{R}$ there is a duality induced by cup product $H^1_T(K, M) \times H^2(K, M^*) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

Write $\chi(M) = \frac{\#H^0(K,M)\#H^0(K,M^*)}{H^1_T(K,M)}$. Show that

$$\chi(M) = |m|_K,$$

where $|m|_K = m$ (resp. $|m|_K = m^2$) if $K$ is real (resp. complex).

7 HW7, due on April 14th

1. Let $m$ be a fixed positive integer. Let $K$ be a number field such that every rational prime $p | m$ is unramified in $K$. Prove that $K$ and $\mathbb{Q}(\mu_m)$ are linearly disjoint.

2. Let $K$ be a number field. Let $\mathbb{A}^\times$ be the group of ideles and $\mathbb{A}_f^\times = \prod_{v<\infty} K_v$ be the group of finite ideles. Give one example $K \neq \mathbb{Q}$ where $K^\times \hookrightarrow \mathbb{A}_f^\times$ (the diagonal embedding) is a discrete subgroup and give one counterexample. Justify your answer.
3. Using class field theory, prove that $\mathbb{Q}(\zeta)$ is the maximal finite abelian extension of $\mathbb{Q}(\sqrt{5})$ that is unramified away from 5 and $\infty$ and has degree prime to 5.

4. Show that, for a number field $K$, the idéle class group $\mathcal{C}_K = \mathbb{I}_K/K^\times$ of $K$ has subgroups of finite index that are not open.

5. (a) Prove that there is a neighborhood of 1 in $\mathbb{C}^\times$ that contains no nontrivial subgroups. Deduce that the kernel of a continuous homomorphism $G \to \mathbb{C}^\times$ for a profinite group $G$ must be an open (hence finite-index) subgroup. 

(b) Let $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ be a continuous homomorphism. Show that for all but finitely many non-archimedean places $v$ of $K$, we have $\chi|_{O_K^\times} = 1$.

(c) A Hecke character for a global field $K$ is a continuous homomorphism $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that $\chi(K^\times) = 1$. (Equivalently, it is a continuous homomorphism $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$). Recall that we have an absolute value map $\| \cdot \| : \mathbb{I}_K = \mathbb{A}_K^\times \to \mathbb{R}_+^\times$ whose kernel is denoted by $\mathbb{I}_K^1$. Note that $\mathbb{I}_K^1/K^\times$ is compact (why?). Show that every Hecke character $\chi$ can be written as $\chi = \chi_0 \cdot \| \cdot \|$ where $\chi_0$ is a Hecke character with valued in the unit circle, and $s \in \mathbb{C}$. Show that the real part of $s$ is uniquely determined.

(d) (Global Langlands correspondence for GL(1)) Prove that the composition with the Artin map $\phi_K : \mathbb{I}_K/K^\times \to G_K^\text{ab}$ defines a bijection between the set of Hecke characters of finite order and the set of continuous homomorphisms $G_K \to \mathbb{C}^\times$.

6. Let $L/K$ be a cyclic extension of number fields, and let $a \in K$. If the image of $a$ in $K_v$ is a norm from $L^v$ for all places $v$, then it is a norm from $L$. (Recall that here $L^v$ denotes $L_w$ for any one place $w$ of $L$ above $v$.) Show that this fails for non-cyclic extension.
8 HW8, due on April 21th

1. Let $K$ be a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$. Show that

$$H^3(\text{Gal}(\overline{K}/K), \overline{K}^\times) = 0.$$ 

What is $H^4(\text{Gal}(\overline{K}/K), \overline{K}^\times)$?

2. Let $K$ be a number field. Show that

$$H^3(\text{Gal}(\overline{K}/K), \mathcal{C}_\overline{K}) = 0$$

where $\mathcal{C}_\overline{K} = \lim_{L/K} \mathcal{C}_L$, for all finite extension $L/K$, and $\mathcal{C}_L$ denotes the idele class group of $L$. What is $H^4(\text{Gal}(\overline{K}/K), \mathcal{C}_\overline{K})$?

3. Let $K$ be a number field. Show that $H^3(\text{Gal}(\overline{K}/K), \overline{K}^\times) = 0$.

(Hint to (1), (2), (3): Class formation and Tate–Nakayama theorem. It is also common to consider $H^i(\text{Gal}(\overline{K}/K, M)$ as the inductive limit $H^i(\text{Gal}(L/K, M_L)$ over all finite Galois $L/K$.)

9 Final exam

1. Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $K_\pi$ be the Lubin-Tate extension of $K$ associated to a uniformizer $\pi$. Let $\pi_1, \pi_2$ be two uniformizers.

   (a) Compute $K_{\pi_1} \cap K_{\pi_2}$, and $K_{\pi_1} K_{\pi_2}$.

   (b) Denote $\Gamma_L = \text{Gal}(\overline{K}/L)$. Can $\Gamma_{K_{\pi_1}}$ and $\Gamma_{K_{\pi_2}}$ generate $\Gamma_K$?

2. (Easy Grunwald-Wang) Let $K$ be a number field, $m = 2^t m', m'$ odd, and $S$ a finite set of primes of $K$. Let $\alpha \in K$ satisfy $\alpha \in (K_v^\times)^m$ for all $v \notin S$. If $K(\zeta_{2^t})/K$ is cyclic (which is certainly satisfied if $t \leq 2$) then $\alpha \in (k^\times)^m$. Show this by following this outline: First, reduce to the case when $m$ is a prime power. Second, if $m = p^r$ ($p$ not necessarily 2), show that if $K(\zeta_{p^r})/K$ is cyclic, then any $\alpha \in (K_v^\times)^{p^r}$ for all $v \notin S$ must be a global $p^r$-th power. Do so in the following way: call $K' = K(\zeta_{p^r})$,
and show that \( \alpha = \beta^p \) for some \( \beta \in K' \) using Kummer theory. Then, by comparing the factorization over \( K \)

\[
x^p - \alpha = \prod_{i=1}^{l} f_i(x)
\]

and the factorization over \( K' \)

\[
x^p - \alpha = \prod_{j=1}^{p'} (x - \beta \zeta_{p^r})
\]

conclude that, choosing a root \( \beta_i \) for each \( f_i \), there are intermediate field \( K \subset K(\beta_i) \subset K' \). Analyze the splitting of \( v \) in these fields for various \( i \) and use that \( K'/K \) is cyclic to conclude. Finally, third, show the theorem–by assumption it suffices to work with \( p \neq 2 \), then take norms from \( K' \) to \( K \) to conclude.

3. ("Counterexample" to easy Grunwald-Wang) Show that 16 is an 8-th power for \( \mathbb{R} \) and all \( \mathbb{Q}_p, p \neq 2 \), but not an 8-th power in \( \mathbb{Q} \). Hint: Show \( \mathbb{Q}(\zeta_8) \) is unramified at all odd \( p \) so for all odd \( p \), \( \mathbb{Q}_p \) contains at least one of \( 1+i, \sqrt{2}, \sqrt{-2} \).

4. Compute the number of degree-\( p \) abelian extensions of a finite extension \( K \) of \( \mathbb{Q}_p \). Your answer will depend on the ramification degree \( c \) of \( K/\mathbb{Q}_p \). (Hint: you may need local Tate duality.)

5. Give an example of global field \( K \) and a Hecke character \( \chi : \mathbb{I}_K \to \mathbb{S}^1 \) such that \( \chi|_{\mathbb{I}_K} \) has infinite order. Here \( \mathbb{I}_K \) is the kernel of the absolute value map

\[
| \cdot | : \mathbb{I}_K = \mathbb{A}^\times \twoheadrightarrow \mathbb{R}^\times_+
\]

\[
(x_v)_{v \in \Sigma_K} \twoheadrightarrow \prod_{v \in \Sigma_L} | x_v |_v
\]

6. Let \( K \) be a finite extension of \( \mathbb{Q}_p \), \( K_\infty/K \) a totally ramified \( \Gamma = \text{Gal}(K_\infty/K) \cong \mathbb{Z}_p \)-extension. We constructed continuous \( K_\infty \)-linear, \( \Gamma \)-equivariant maps

\[
\pi_n : \widehat{K}_\infty \to K_n,
\]
by \( \pi_n(x) = p^{-m} \text{tr}_{K_{n+m}/K_n}(x) \) when \( x \in K_{n+m} \), and extending by continuity. Show that for \( x \in \hat{K}_\infty \) we have
\[
x = \underset{n}{\lim} \pi_n(x).
\]

7. Let \( K'/K \) be a finite extension of number fields. Denote by \( H_K, H_{K'} \) the Hilbert Class Fields of \( K \) and \( K' \) respectively, i.e. the maximal unramified extensions of each (at all places, including both finite and infinite; recall that we say that an archimedean local field extension \( F'/F \) is ramified only when \( F' = \mathbb{C} \) and \( F = \mathbb{R} \)).

(a) Show that the norm map \( Cl_{K'} \rightarrow Cl_K \) by \( N[A] := [N_{K'/K}A] \) is a well-defined homomorphism between ideal class groups. By checking on classes of prime ideals prove that the diagram
\[
\begin{array}{ccc}
Cl_{K'} & \longrightarrow & Gal(H_{K'}/K') \\
\downarrow \scriptstyle N & & \downarrow \\
Cl_K & \longrightarrow & Gal(H_K/K)
\end{array}
\]

commutes, where the horizontal maps are the natural isomorphisms (described by Frobenius elements on classes of primes, i.e., induced by the Artin homomorphism) and the right vertical map is the natural restriction map induced by the inclusion \( H_K \hookrightarrow H_{K'} \).

(b) Let \( L \) be the maximal subextension of \( K' \) that is abelian and everywhere (including infinite places) unramified over \( K \). Show that the norm map between ideal class groups is surjective (and hence \( h_K|h_{K'} \)) when \( L = \hat{K} \).

(c) Prove that if \( K'/K \) is totally ramified at some place (perhaps archimedean in case \( [K' : K] = 2 \)) then \( h_K|h_{K'} \).