Stability and the structure of the derived category of coherent sheaves on irreducible curves of genus one.

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(with Igor Burban)

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1 Relative Fourier-Mukai transforms for singular genus one fibrations
   • The setting
   • The result
   • The proof
   • Duality

2 The structure of $\mathcal{D}^b_{\text{coh}}(E)$ on irreducible singular curves of genus one
   • Comparison smooth – singular, I
   • Standard stability
   • Stability conditions
   • Comparison smooth – singular, II
1. Relative Fourier-Mukai transforms for singular genus one fibrations
   - The setting
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2. The structure of $\mathcal{D}^b_{\text{coh}}(E)$ on irreducible singular curves of genus one
   - Comparison smooth – singular, I
   - Standard stability
   - Stability conditions
   - Comparison smooth – singular, II
Motivation

\[ X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X \]

\[ \mathcal{P} \in D^b_{coh}(X \times X) \]

Mukai: \( R^{\pi_2*}(\mathcal{P} \otimes \pi_1^*(\cdot)) \)

FMT \iff equivalence.

- moduli of vector bundles, \( D^b_{coh}(X) \);
- In higher-dimensional birational geometry, \( D^b_{coh}(X) \) is used to study MMP.
- Most results about FMT assume smoothness, but MMP leads to singular varieties.
Elliptic fibrations

$k$ algebraically closed, characteristic zero, $X, S$ schemes
- reduced, connected;
- separated, of finite type over $k$.

$q : X \longrightarrow S$
- projective, flat;
- with section $\sigma : S \to X$, $\Sigma := \sigma(S)$
- fibers are integral Gorenstein curves of arithmetic genus one.

We allow $X$ to be singular.
Neither $X$ nor $S$ needs to be Gorenstein.
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We allow $X$ to be singular.
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The result

\[ X \times_S X \xrightarrow{\pi_1} X \]
\[ \downarrow \pi_2 \] \[ \downarrow q \]
\[ X \xrightarrow{q} S \]

\[ \mathcal{P} := \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_X(\Sigma) \otimes \pi_2^* \mathcal{O}_X(\Sigma) \]

\[ \text{FM}_p(\mathcal{E}^\bullet) := \mathcal{R}\pi_2^*(\mathcal{P} \overset{L}{\otimes} \pi_1^*(\mathcal{E}^\bullet)) \]

**Theorem**

\[ \text{FM}_p : D^-_{\text{coh}}(X) \rightarrow D^-_{\text{coh}}(X) \text{ equivalence.} \]

Same is true for \( D^b_{\text{coh}}, D^+_{\text{coh}}, D_{\text{coh}} \).
The result

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$$\pi_2 \downarrow \quad \downarrow q$$

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$$\mathcal{P} := \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_X(\Sigma) \otimes \pi_2^* \mathcal{O}_X(\Sigma)$$

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Same is true for $D^b_{\text{coh}}, D^+_{\text{coh}}, D_{\text{coh}}$. 
Proof – preliminaries

Starting point: \( S = \text{Spec}(k), \ X \) smooth elliptic curve

Mukai’s result: \([1] \circ \text{FM}_P \circ \text{FM}_P \cong (-1)^*\)

where \((-1) : X \to X\) is “taking the inverse”.

Theorem (Burban, K.)

\([1] \circ \text{FM}_P \circ \text{FM}_P \cong i^*\)

Assumptions: \( S = \text{Spec}(k) \) and \( i : X \to X \) is “taking the inverse” on the smooth locus of \( X \).

Involution \( i : X \to X \) exists for elliptic fibrations with section.
Proof – preliminaries

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Relative FMT – singular case
$D^b_{coh}(E)$ for irreducible singular curves of genus one

The setting
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The proof
Duality

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Proof

Idea: understand
\[ [1] \circ i^* \circ \text{FM}_P \circ \text{FM}_P = \text{FM}_{Q^\bullet} \]
by restricting to fibers.

Lemma

With \( P_s^\bullet := L(j_s \times j_s)^* P^\bullet \), we have \( j_s^* \circ \text{FM}_{P_s^\bullet} \cong \text{FM}_{P^\bullet} \circ j_s^* \).

\[ \Rightarrow \quad \text{FM}_{Q^\bullet} \circ j_s^* \cong j_s^* \]
\[ \Rightarrow \quad \forall x \in X \quad \text{FM}_{Q^\bullet}(k(x)) \cong k(x) \]
Bridgeland lemma \( \Rightarrow \quad Q^\bullet \cong \delta_* L, \quad L \in \text{Pic}(X) \)
\[ \Rightarrow \quad \text{FM}_P \circ \text{FM}_P \cong [-1] \circ i^* \circ (q^* N \otimes \cdot) \]
\[ \Rightarrow \quad \text{FM}_P \text{ is an equivalence.} \]
Proof

Idea: understand

\[ [1] \circ i^* \circ FM_P \circ FM_P = FM_Q^\bullet \]

by restricting to fibers.

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With \( P_s^\bullet := L(j_s \times j_s)^* P^\bullet \), we have \( j_s^* \circ FM_{P_s^\bullet} \cong FM_{P^\bullet} \circ j_s^* \).

\[ \Rightarrow \quad FM_{Q^\bullet} \circ j_s^* \cong j_s^* \]

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Bridgeland lemma \( \Rightarrow \quad Q^\bullet \cong \delta_* \mathcal{L}, \mathcal{L} \in \text{Pic}(X) \)

\[ \Rightarrow \quad FM_P \circ FM_P \cong [-1] \circ i^* \circ (q^* \mathcal{N} \otimes \cdot) \]

\[ \Rightarrow \quad FM_P \text{ is an equivalence.} \]
Duality

- Grothendieck-Verdier duality gives:
  \[
  R\mathcal{H}om_X(\mathcal{F}M_P(\cdot), \mathcal{O}_X)
  \]
  and
  \[
  \mathcal{F}M_P(R\mathcal{H}om_X(\cdot, \mathcal{O}_X))
  \]

  are equal up to \([1] \circ i^* \circ (q^* \mathcal{N} \otimes \cdot)\);

- \(S\) Gorenstein, then \(\mathcal{D}_X := R\mathcal{H}om_X(\cdot, \mathcal{O}_X)\) is a dualizing functor;

- \(S = \text{Spec}(k)\), then

\[
\mathcal{D}_X \circ \mathcal{F}M_P \cong [1] \circ i^* \circ \mathcal{F}M_P \circ \mathcal{D}_X.
\]
Relative Fourier-Mukai transforms for singular genus one fibrations

- The setting
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The structure of $D^b_{coh}(E)$ on irreducible singular curves of genus one

- Comparison smooth – singular, I
- Standard stability
- Stability conditions
- Comparison smooth – singular, II
The smooth case

Let $E$ be a smooth elliptic curve. For coherent sheaves, we have:

\[
\begin{align*}
\text{locally free} & \iff \text{torsion free} \\
\text{stable} & \iff \text{simple} \\
\text{indecomposable} & \implies \text{semi-stable}
\end{align*}
\]

- homological dimension of $\text{Coh}(E)$ is 1, i.e. $\text{Ext}^i(F, G) = 0 \quad \forall \ i > 1$;
- simple structure of $D^b_{\text{coh}}(E)$: any object is the direct sum of its cohomology sheaves;
- important tool: Serre Duality  
\[
\forall F, G \in \text{Coh}(E) : \quad \text{Ext}^i(F, G) \cong \text{Ext}^{1-i}(G, F)^*
\]
The smooth case

Let $E$ be a smooth elliptic curve. For coherent sheaves, we have:

- locally free $\iff$ torsion free
- stable $\iff$ simple
- indecomposable $\implies$ semi-stable

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  \[ \forall \ F, G \in \text{Coh}(E) : \quad \text{Ext}^i(F, G) \cong \text{Ext}^{1-i}(G, F)^* \]
### Comparison

<table>
<thead>
<tr>
<th></th>
<th>smooth</th>
<th>singular</th>
</tr>
</thead>
<tbody>
<tr>
<td>homological dim of $\text{Coh}(E)$</td>
<td>1</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Serre duality holds</td>
<td>in general</td>
<td>if one object is perfect</td>
</tr>
<tr>
<td>torsion free implies locally free</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>indecomposable coherent sheaves are semi-stable</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>any indecomposable complex is isomorphic to a shift of a sheaf</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

$D^b_{\text{coh}}(E)$ for irreducible singular curves of genus one

Comparison smooth – singular, I
Standard stability
Stability conditions
Comparison smooth – singular, II
Stability and HNF

**E** irreducible singular curve of arithmetic genus one over \( \mathbf{k} \), algebraically closed, characteristic zero. \( 0 \neq \mathcal{F} \in \text{Coh}(E) \)

- **slope** \( \mu(\mathcal{F}) = \text{deg}(\mathcal{F})/\text{rk}(\mathcal{F}) \)
- **phase** \( \varphi(\mathcal{F}) \in (0, 1] \) such that
  \[-\text{deg}(\mathcal{F}) + i \cdot \text{rk}(\mathcal{F}) \in \mathbb{R}_{>0} \cdot \exp(i\pi\varphi(\mathcal{F})) \]
- \( \varphi(\mathcal{O}) = 1/2, \quad \varphi(\mathbf{k}(x)) = 1 \)
- \( \mathcal{F} \) semi-stable \( \iff \forall 0 \neq \mathcal{G} \subset \mathcal{F} : \varphi(\mathcal{G}) \leq \varphi(\mathcal{F}) \).

Any \( \mathcal{F} \in \text{Coh}(E) \) has a HNF

\[
0 \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F}
\]

such that the factors \( \mathcal{A}_i = \mathcal{F}_i/\mathcal{F}_{i+1} \) are semi-stable and

\[
\varphi(\mathcal{A}_n) > \varphi(\mathcal{A}_{n-1}) > \ldots > \varphi(\mathcal{A}_0).
\]
Stability and HNF

**E** irreducible singular curve of arithmetic genus one over \( k \), algebraically closed, characteristic zero. \( 0 \neq \mathcal{F} \in \text{Coh}(E) \)

- **slope** \( \mu(\mathcal{F}) = \deg(\mathcal{F})/\text{rk}(\mathcal{F}) \)
- **phase** \( \varphi(\mathcal{F}) \in (0, 1] \) such that \(-\deg(\mathcal{F}) + i \cdot \text{rk}(\mathcal{F}) \in \mathbb{R}_{>0} \cdot \exp(i \pi \varphi(\mathcal{F}))\)
- \( \varphi(\mathcal{O}) = 1/2, \quad \varphi(k(x)) = 1 \)
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Relative FMT – singular case

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Stability and HNF

$E$ irreducible singular curve of arithmetic genus one over $k$, algebraically closed, characteristic zero. $0 \neq \mathcal{F} \in \text{Coh}(E)$

- slope $\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\text{rk}(\mathcal{F})}$
- phase $\varphi(\mathcal{F}) \in (0, 1]$ such that
  $\frac{\deg(\mathcal{F})}{\text{rk}(\mathcal{F})} - \text{deg}(\mathcal{F}) + i \cdot \text{rk}(\mathcal{F}) \in \mathbb{R}_{>0} \cdot \exp(i\pi \varphi(\mathcal{F}))$
- $\varphi(\mathcal{O}) = 1/2$, $\varphi(k(x)) = 1$
- $\mathcal{F}$ semi-stable $\iff \forall 0 \neq \mathcal{G} \subset \mathcal{F}: \varphi(\mathcal{G}) \leq \varphi(\mathcal{F})$.

Any $\mathcal{F} \in \text{Coh}(E)$ has a HNF

$$0 \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F}$$

such that the factors $\mathcal{A}_i = \mathcal{F}_i / \mathcal{F}_{i+1}$ are semi-stable and

$$\varphi(\mathcal{A}_n) > \varphi(\mathcal{A}_{n-1}) > \ldots > \varphi(\mathcal{A}_0).$$
Stability on $D^b_{\text{coh}}(E)$

- $\varphi(\mathcal{F}[n]) := \varphi(\mathcal{F}) + n$ for $n \in \mathbb{Z}$ and $\mathcal{F} \in \text{Coh}(E)$.
- Slicing $P(\varphi) = \{ \text{semi-stable sheaves of phase } \varphi \}$

**Theorem (Bridgeland; GKR)**

$0 \neq X \in D^b_{\text{coh}}(E)$ has a HNF, unique up to isomorphism,

$$
0 \to F_nX \to \cdots \to F_2X \to F_1X \to F_0X = X
$$

with $0 \neq A_i \in P(\varphi_i)$ and $\varphi_+(X) = \varphi_n > \varphi_{n-1} > \cdots > \varphi_1 > \varphi_0 = \varphi_-(X)$. 

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Stability and the derived category on singular curves
Stability on $D^{b}_{\text{coh}}(E)$

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with $0 \neq A_{i} \in P(\varphi_{i})$ and $\varphi_{+}(X) = \varphi_{n} > \varphi_{n-1} > \cdots > \varphi_{1} > \varphi_{0} = \varphi_{-}(X)$. 

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Stability and the derived category on singular curves
general:  \( \varphi_-(X) > \varphi_+(Y) \implies \text{Hom}(X, Y) = 0 \)

curve case:
\( \varphi_-(X) < \varphi_+(Y) < \varphi_-(X) + 1 \implies \text{Hom}(X, Y) \neq 0 \)

The main tool to prove this are Seidel-Thomas twists:

\[
T_E : D^b_{\text{coh}}(E) \to D^b_{\text{coh}}(E)
\]

given by  \( \text{RHom}(E, F) \otimes E \to F \to T_E(F) \xrightarrow{+} \)

\( T_E \) is an equivalence, if \( E \) is spherical, i.e. perfect and

\[
\text{Hom}(E, E[i]) = \begin{cases} 
  k & \text{if } i = 0, 1 \\
  0 & \text{otherwise.}
\end{cases}
\]
Relative FMT – singular case

$D^b_{\text{coh}}(E)$ for irreducible singular curves of genus one

- **general:** $\varphi_-(X) > \varphi_+(Y) \implies \text{Hom}(X, Y) = 0$
- **curve case:**
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The main tool to prove this are Seidel-Thomas twists:

$$T_E : D^b_{\text{coh}}(E) \to D^b_{\text{coh}}(E)$$

- given by $R\text{Hom}(E, F) \otimes E \to F \to T_E(F) \xrightarrow{\dagger}$
- $T_E$ is an equivalence, if $E$ is spherical, i.e. perfect and

$$\text{Hom}(E, E[i]) = \begin{cases} k & \text{if } i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$
$D_{\text{coh}}^b(E)$ for irreducible singular curves of genus one

\[ \mathbb{F} = T_{k(p_0)} T \circ T_{k(p_0)} \]

$\mathbb{F} \cong \text{FMT with kernel } \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}(p_0) \otimes \pi_2^* \mathcal{O}(p_0)[1]$

**Theorem (Bruzzo et al.)**

$\mathbb{F}$ preserves semi-stability.

Our proof uses

- the degree zero case (math.AG/0401437);
- $\mathbb{F} \mathbb{F} = i^*[1]$ (math.AG/0401437);
- $\mathbb{D} \mathbb{F} = i^* \mathbb{F} \mathbb{D}[1]$ (math.AG/0410349).
Relative FMT – singular case

$D^b_{\text{coh}}(E)$ for irreducible singular curves of genus one

Comparison smooth – singular, I
Standard stability
Stability conditions
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\mathbb{F} = T_{k(p_0)} T \circ T_{k(p_0)}
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\[\mathbb{F} \cong \text{FMT with kernel } \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}(p_0) \otimes \pi_2^* \mathcal{O}(p_0)[1]\]

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- $\mathbb{D} \mathbb{F} = i^* \mathbb{F} \mathbb{D}[1]$ (math.AG/0410349).
\[ F = T_{k(p_0)} T \mathcal{O} T_{k(p_0)} \]

\[ F \cong \text{FMT with kernel } \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}(p_0) \otimes \pi_2^* \mathcal{O}(p_0)[1] \]

**Theorem (Bruzzo et al.)**

\( F \) preserves semi-stability.

Our proof uses

- the degree zero case (math.AG/0401437);
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Relative FMT – singular case

\( D^b_{\text{coh}}(E) \) for irreducible singular curves of genus one

**Comparison smooth – singular, I**

**Standard stability**

**Stability conditions**

**Comparison smooth – singular, II**

\[ \widetilde{\text{SL}(2, \mathbb{Z})} - \text{action} \]

\( \widetilde{\text{SL}(2, \mathbb{Z})} = \langle A, B, T \mid ABA = BAB, (AB)^6 = T^2, AT = TA, BT = TB \rangle \), a central extension of \( \text{SL}(2, \mathbb{Z}) \), acts on \( D^b_{\text{coh}}(E) \) by

\[ A \mapsto T_0, \quad B \mapsto T_{k(p_0)}, \quad T \mapsto [1]. \]

**Corollary**

\( \widetilde{\text{SL}(2, \mathbb{Z})} \) acts transitively on the set of non-zero slices \( \text{P}(\varphi) \).

\[
\begin{align*}
\text{P}(\varphi) & \xrightarrow{\sim} \text{P}(1) \xrightarrow{\text{coherent torsion sheaves}} \\
\text{P}(\varphi)^s & \xrightarrow{\sim} \text{P}(1)^s \xrightarrow{\text{k(x) \mid x \in E}} \text{stable objects}
\end{align*}
\]
Structure of $\mathbb{P}(\varphi)$

- Objects in $\mathbb{P}(\varphi)$ have JHF with stable JH-factors;
- Indecomposable objects have a single JH-factor;
- We call the unique non-perfect element in $\mathbb{P}(\varphi)^s$ the *extreme* stable element of phase $\varphi$;
- E.g. $k(s) \in \mathbb{P}(1)^s$, if $s \in \mathbb{E}$ is the singular point;
- stable objects are either perfect or extreme;
- $\mathbb{F}(\mathbb{P}(\varphi)) = \mathbb{P}(\varphi + \frac{1}{2})$. 

<table>
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<tr>
<th></th>
<th>$\text{Coh}_E[1]$</th>
<th>$\text{Coh}_E$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>2</td>
<td>1</td>
<td>$-1$</td>
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Structure of $P(\varphi)$

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- We call the unique non-perfect element in $P(\varphi)^{s}$ the *extreme* stable element of phase $\varphi$;
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$$
\begin{array}{|c|c|c|c|}
\hline
 & \text{Coh}_E[1] & \text{Coh}_E & \text{Coh}_E[-1] \\
\hline
2 & 1 & t & 0 & -1 \\
\hline
\end{array}
$$
The shadow of an indecomposable object is the set of all JH-factors of HN-factors, connected by line segments.

Theorem

If $X$ is indecomposable and not semi-stable, then all direct summands of its HN-factors are non-perfect.
Shadows

The **shadow** of an indecomposable object is the set of all JH-factors of HN-factors, connected by line segments.

\[ X_3 \quad \text{Coh}_E[1] \quad X_4 \quad X_5 \quad \text{Coh}_E[\overline{-1}] \]

\[ 2 \quad 1 \quad 0 \quad -1 \]

**Theorem**

*If* \( X \) *is indecomposable and not semi-stable, then all direct summands of its HN-factors are non-perfect.*
Indecomposable Objects

Corollary

There exist four types of indecomposable objects in $\text{Coh}(\mathbf{E})$:

1. semi-stable with perfect JH-factor;
2. semi-stable, perfect but with extreme JH-factor;
3. semi-stable and not perfect;
4. not semi-stable, all HN-factors without a perfect summand.
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\[ \begin{array}{c|c|c|c|c} \hline \\
2 & 1 & 0 & -1 \\
\hline \\
X_1 & & & \\
\hline \\
X_1' & & & \\
\hline \end{array} \]
Indecomposable Objects

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Stability Conditions – Definition

**Definition (Bridgeland)**

\((W, R)\) stability condition

- \(W : K(E) \to \mathbb{C}\) homomorphism of groups;
- \(R\) is a compatible slicing, i.e.
  - \(R(t + 1) = R(t)[1]\),
  - \(\text{Hom}(A_1, A_2) = 0\), if \(t_1 > t_2\) and \(A_i \in R(t_i)\),
  - \(0 \neq X \in D^b_{\text{coh}}(E)\) has a HNF with \(A_i \in R(\varphi_i)\),
  - \(W(A) \in \mathbb{R}_{>0} \exp(i\pi t)\), if \(A \in R(t)\).
Theorem

\( \tilde{\mathbb{GL}}^+ (2, \mathbb{R}) \) acts simply transitive on \( \text{Stab}(E) \).

\( \tilde{\mathbb{GL}}^+ (2, \mathbb{R}) \) = all pairs \((A, f)\) with \( A \in \mathbb{GL}^+(2, \mathbb{R}) \) and \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \) is strictly increasing, \( f(t + 1) = f(t) + 1 \) and \( A \) and \( f \) induce the same mapping on \( (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}^* = S^1 = \mathbb{R}/2\mathbb{Z} \).

\[
(A, f) \cdot (W, R) := (A^{-1} \circ W, R \circ f)
\]

Main tool: we describe all \( t \)-structures on \( D^b_{\text{coh}}(E) \).
Common features

1. $F \in \text{Coh}(E)$ is stable if and only if $\text{End}(F) \cong k$.
2. Any spherical object is a shift of a stable vector bundle or of a structure sheaf $k(x)$ at a smooth point $x \in E$.
3. The category of semi-stable sheaves of a fixed slope is equivalent to the category of coherent torsion sheaves. Such an equivalence is induced by an auto-equivalence of $D^b_{\text{coh}}(E)$.
4. With $\text{Aut}^0 = \langle \text{Aut}(E), \text{Pic}^0(E), [2] \rangle$, there is an exact sequence of groups

$$0 \rightarrow \text{Aut}^0 \rightarrow \text{Aut}(D^b_{\text{coh}}(E)) \rightarrow \text{SL}(2, \mathbb{Z}) \rightarrow 0.$$

5. $\widetilde{\text{GL}}^+(2, \mathbb{R})$ acts transitively on $\text{Stab}(E)$.
6. $\text{Stab}(E)/\text{Aut}(D^b_{\text{coh}}(E)) \cong \text{GL}^+(2, \mathbb{R})/\text{SL}(2, \mathbb{Z})$. 