1. State the theorem on the existence and uniqueness of the determinant.

2. (a) If $A$ is a square matrix with real entries, prove that $AA^T$ is symmetric.
   
   (b) Show that the symmetric matrix \(
   \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 3 \\
   0 & 3 & 1
   \end{pmatrix}
   \) cannot be expressed as $AA^T$ for any square $A$ with real entries.

3. Let $A$ be a square matrix with complex entries.
   
   (a) Prove that $\det A^* = \overline{\det A}$ by reduction by minors.
   
   (b) If $\lambda$ is an eigenvalue of $A$, show that $\overline{\lambda}$ is an eigenvalue of $A^*$.

4. Let $V \subset \mathcal{F}([-1, 1], \mathbb{R})$ be the space of continuous functions $[-1, 1] \to \mathbb{R}$, with Euclidean inner product given by
   
   \[
   \langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx.
   \]
   
   Find an orthonormal basis for the subspace spanned by $1, x, x^2$.
   
   (Apply the Gram-Schmidt process to this subspace.)

5. (a) Find the inverse of the matrix $B = \begin{pmatrix}
   0 & 1 & 3 \\
   1 & 0 & 1 \\
   2 & 0 & 1
   \end{pmatrix}$.
   
   (b) If $\vec{c} = \begin{pmatrix}
   1 \\
   2 \\
   5
   \end{pmatrix}$, find all solutions $\vec{x}$ of the inhomogeneous system $B\vec{x} = \vec{c}$.

6. Let $A$ be a square matrix with real entries. If $\lambda$ is a complex eigenvalue of $A$, prove that $\overline{\lambda}$ is also an eigenvalue of $A$.

7. (a) If $X, Y$ are vectors in a real Euclidean space, prove that $\langle X, Y \rangle = 0$ if and only if $\|X - Y\| = \|X + Y\|$.
   
   (b) Give a counterexample showing that this is false in a complex Hermitian space.

   **ANSWERS OVERLEAF...**
Answers to Practice Midterm

1. There exists an unique det : $M_{n \times n} \to \mathbb{R}$ which, when expressed as a function det($\vec{a}_1, \ldots, \vec{a}_n$) of the rows $\vec{a}_1, \ldots, \vec{a}_n$ of $A \in M_{n \times n}$, satisfies the following:
   
   D1: For any $i \in \{1, \ldots, n\}$ and any $c \in \mathbb{R}$, det($\vec{a}_1, \ldots, c\vec{a}_i, \ldots, \vec{a}_n$) = $c$ det($\vec{a}_1, \ldots, \vec{a}_n$).
   
   D2: For any $i \in \{1, \ldots, n\}$ and any $\vec{b}_i \in \mathbb{R}^n$,
   
   $\det(\vec{a}_1, \ldots, \vec{a}_i + \vec{b}_i, \ldots, \vec{a}_n) = \det(\vec{a}_1, \ldots, \vec{a}_i, \ldots, \vec{a}_n) + \det(\vec{a}_1, \ldots, \vec{b}_i, \ldots, \vec{a}_n)$.
   
   D3: If two distinct rows are equal, say $\vec{a}_i = \vec{a}_j$ for $i \neq j$, then det($\vec{a}_1, \ldots, \vec{a}_n$) = 0.
   
   D4: $\det I_n = 1$.

2. (a) $(AA^T)^T = (A^T)^T A^T = AA^T$, so this is symmetric.
   
   (b) $\det(AA^T) = \det A \det A^T = (\det A)^2 \geq 0$, but this matrix has determinant $-8$.

3. (a) $A^* = \overline{A^T}$ and $\det A^T = \det A$, so it suffices to show det $\overline{A} = \overline{\det A}$. Proof by induction on $n$, where $A$ is $n \times n$. Obvious for $n = 1$. If true for $n - 1$, $\det \overline{A} = \sum_i (-1)^i \overline{a_{i1}} \det \overline{A_{i1}} = \sum_i (-1)^i \overline{a_{i1}} \det A_{i1} = \det A$, the 2nd equality by induction, the 3rd since conjugation commutes with taking sums and products.
   
   (b) If $\det(\lambda I - A) = 0$, then by (a), $\det(\overline{\lambda} I - A^*) = \det(\overline{\lambda} I^* - A^*) = \det(\lambda I - A^*) = 0$.

4. $\|1\|^2 = \langle 1, 1 \rangle = 2$, so let $u_1 = 1/\|1\| = 1/\sqrt{2}$. Then $\langle u_1, x \rangle = 0$, so let $u_2 = x/\|x\| = x/\sqrt{2}/3 = \sqrt{3}x/\sqrt{2}$. Finally, $\langle u_1, x^2 \rangle = \sqrt{2}/3$, and $\langle u_2, x^2 \rangle = 0$, so let $u_3$ be $x^2 - \sqrt{2}/3 u_1$ divided by its own norm, which is $(3\sqrt{5})/(2\sqrt{2})(x^2 - 1/3)$.

5. (a) Elementary row operations on the augmented matrix $(B|I)$ lead to $(I|B^{-1})$ where
   
   $B^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -6 & 3 \\ 0 & 2 & -1 \end{pmatrix}$.
   
   (On a real exam, you must show your work! Also, it’s wise to check $BB^{-1} = I$.)
   
   (b) Multiplying both sides by $B^{-1}$ shows that the unique solution is $B^{-1} \vec{c} = (3, 4, -1)$.

6. Proof 1: let $\chi(t) = \det(tI - A)$ be the characteristic polynomial of $A$. Then clearly $\chi$ has real coefficients, that is, $\chi(t) = c_0 + c_1 t + c_2 t^2 + \cdots + t^n$ with $c_i \in \mathbb{R}$. Now $\lambda$ is an eigenvalue if and only if $\chi(\lambda) = 0$, but then $\chi(\overline{\lambda}) = c_0 + c_1 \overline{\lambda} + c_2 \overline{\lambda}^2 + \cdots + \overline{\lambda}^n = \overline{c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots + \lambda^n} = 0 = 0$, so $\overline{\lambda}$ is also an eigenvalue.

   Proof 2: say $X = (x_1, \ldots, x_n)$ is the eigenvector, $AX = \lambda X$. Let $\overline{X} = (\overline{x_1}, \ldots, \overline{x_n})$. Then since $A$ is real, $A\overline{X} = \overline{AX} = \overline{\lambda X} = \overline{\lambda} \overline{X}$, so $\overline{X}$ is an eigenvector with eigenvalue $\overline{\lambda}$.

7. (a) $\|X - Y\| = \|X + Y\| \iff \langle X - Y, X - Y \rangle = \langle X + Y, X + Y \rangle \iff \langle X, X \rangle - \langle Y, Y \rangle + \langle Y, X \rangle + \langle X, Y \rangle = 0 = 2\langle Y, X \rangle + 2\langle X, Y \rangle = 4\langle X, Y \rangle \iff \langle X, Y \rangle = 0$.
   
   (b) In $\mathbb{C}^2$ with the standard Hermitian dot product, let $X = (i, 0)$ and $Y = (1, 0)$. Then $\langle X, Y \rangle = i$ but $\|X - Y\| = \|X + Y\| = \sqrt{2}$. (Note: the proof in (a) no longer works since we no longer have $\langle X, Y \rangle = \langle Y, X \rangle$.)