Mathematics V1208y
Honors Mathematics IV
Final Examination
May 9, 2016

If a region has graph type; or if a parametrization is counterclockwise; or if a reparametrization is forward or backward, just say so. You don’t have to prove it. Good luck!

PART A: True/False. Decide whether the given statement is true or false, and give a brief reason for your answer (sketch of proof or counterexample). 4 points each.

1. If an invertible matrix $A$ is skew-symmetric (i.e. $A_{ji} = -A_{ij}$), then so is its inverse.
2. The eigenvalues of a matrix with rational entries are rational.
3. If the nonzero vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$ are nonzero and all orthogonal, then they are linearly independent.
4. Any linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable.
5. There exists a scalar field $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ whose partial derivatives are all continuous, but which is not itself continuous.
6. If $U \subseteq \mathbb{R}^2$ is the union of the two rectangles $(1, 2) \times (-1, 3)$ and $(-1, 3) \times (1, 2)$, then the vector field $F(x, y) = (x/(x^2 + y^2), y/(x^2 + y^2))$ is a gradient on $U$.

PART B: Shorter proofs and computations. 7 points each.

7. If $A$ is a Hermitian matrix and all the eigenvalues of $A - I$ are imaginary (that is, real multiples of $i$), prove that $A = I$.
8. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be any continuous scalar field. Show that $U = \{ x \in \mathbb{R}^n | f(x) \neq 0 \}$ is an open set.
9. Use the chain rule to express $\frac{\partial g}{\partial t}$ in terms of partial derivatives of $f(x, y, z)$ if $g(s, t) = f(s^2 - t^2, s^2 + t^2, st)$, and $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a $C^1$ function.
10. Give a careful definition of the integral of a scalar field with respect to arclength.
11. Show that if $f$ is a scalar field and $G$ is a vector field on $\mathbb{R}^3$, then $\nabla \cdot (f^2 G) = 2f \nabla f \cdot G + f^2 \nabla \cdot G$. Be sure you understand what each term means!
12. Using the divergence theorem, compute the surface integral $\int_S F \cdot d\mathbf{r}^2$, where $S$ is the unit upper hemisphere in $\mathbb{R}^3$, and $F(x, y, z) = (x, y, 0)$. 

PART C: Longer proofs and computations. 10 points each.

13. Suppose \( f: \mathbb{R}^2 \to \mathbb{R} \) is continuous, and let \( k(t) = \int_0^1 \int_0^{t^2} f(x,y) \, dx \, dy \). Prove that \( k \) is differentiable and express its derivative in terms of a single integral. State clearly what theorems you are using. (Hint: recall we proved that \( \int_0^1 f(x,y) \, dy \) is a continuous function of \( x \).)

14. Let \( R \) be the rectangle in \( \mathbb{R}^2 \) with vertices at \((0,2), (2,0), (4,6), \) and \((6,4)\). Use the transformation formula to express \( \iint_R xy \, dx \, dy \) as a double integral with constant limits of integration, and evaluate it.

15. Let \( a = (1,0,0) \in \mathbb{R}^3 \), and let \( H \) be the vector field given by \( H(x) = a \times x \).
   (a) Compute the curl of \( H \).
   (b) If \( D \) is the unit disk in the plane, and \( r: D \to S \) is the parametric surface with \( r(u,v) = (u, v, (1 - u^2 - v^2) e^{7u^2v} \cos v) \), compute \( \iint_S a \cdot dr^2 \).
   (c) (Optional) Interpret in terms of fish.

16. Let \( g \) and \( h: \mathbb{R} \to \mathbb{R} \) be increasing \( C^1 \) functions, and let \( F \) be the vector field on \( \mathbb{R}^2 \) given by \( F(x,y) = (g(x) - h(y), h(x) - g(y)) \). Use Green’s theorem to show that, if \( C_r \) is the circle of radius \( r \), then the (counterclockwise) line integral \( \oint_{C_r} F \cdot d\gamma \) is an increasing function of \( r \).

17. Let \( F \) be a \( C^1 \) vector field on \( \mathbb{R}^3 \). Let \( S_r \) be the sphere of radius \( r \) centered at \( 0 \), and let \( n \) be the unit outward normal. Suppose there exist constants \( a,b \) such that for all \( r > 1 \),
   \[ \iint_{S_r} F \cdot dr^2 = ar + b. \]
   (a) What conditions must \( a,b \) satisfy if \( \text{div} F(x) = 0 \) whenever \( ||x|| \geq 1 \)?
   (b) What conditions must \( a,b \) satisfy if there exists a vector field \( G \) on \( \mathbb{R}^3 \) so that \( \text{curl} G(x) = F(x) \) whenever \( ||x|| \geq 1 \)?