1. Apostol §1.17 (p. 30) 2, 3, 4.

*2. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 1, 1)$, $(-1, 4, 4, -1)$, and $(4, -2, 2, 0)$.

3. Apostol §5.5 (pp. 118–120) 3, 4, 5, 6.

4. Apostol §5.11 (pp. 124–126) 1, 2, 6, 8, *13, 14.
   Hint for 13: If $v, w \in \mathbb{R}^n$ are regarded as $n \times 1$ matrices, then $v \cdot w = v^t w$.

*5. Let $V$ be a finite-dimensional Euclidean or Hermitian space, $U \subset V$ any subspace. Show that $\dim U + \dim U^\perp = \dim V$.

*6. (20 pts) Let $U, V, W$ be finite-dimensional Euclidean or Hermitian spaces, $S : U \to V$ and $T : V \to W$ linear maps. Show that:
   
   (a) $T^\circ\circ = T$. (Choosing ONBs is perfectly legitimate, but working directly from the definition is in better taste.)
   
   (b) $\ker T^* = (\text{im } T)^\perp$.
   
   (c) $\ker T = (\text{im } T^*)^\perp$.
   
   (d) $\text{rank } T^* = \text{rank } T$.
   
   (e) Use the above to give an alternate proof that the row-rank of a real square matrix equals its column-rank.
   
   (f) $(TS)^* = S^* T^*$.
   
   (g) If $T$ is invertible, then so is $T^*$, and $(T^*)^{-1} = (T^{-1})^*$.
**7.** (20 pts) Prove the *finite-dimensional spectral theorem*:

Let \( V \) be a finite-dimensional Hermitian space and \( T : V \rightarrow V \) a linear map. Then \( V \) has an orthonormal basis of \( T \)-eigenvectors if and only if \( T \) is normal (i.e. \( TT^* = T^*T \)).

Step 1. Prove the “only if” part by considering a diagonal matrix representation of \( T \).

Step 2. Show that *any* linear map \( T : V \rightarrow V \) can be written \( T = H + iK \), where \( H \) and \( K \) are Hermitian.

Step 3. If \( T \) is normal, show that \( H \) and \( K \) commute in the above description.

Step 4. Let \( \lambda \) be an eigenvalue of \( H \), and \( E_{\lambda} \) the corresponding eigenspace. Show that \( K \) maps \( E_{\lambda} \) into itself, and so by the Hermitian case of the spectral theorem, \( E_{\lambda} \) has an orthonormal basis of \( K \)-eigenvectors.

Step 5. Show that two commuting self-adjoint linear maps can be “simultaneously diagonalized,” i.e. there exists an orthonormal basis of \( V \) consisting of eigenvectors for both \( H \) and \( K \).

Step 6. Show \( V \) has an orthonormal basis of \( T \)-eigenvectors.

**8.** A linear map \( T : V \rightarrow V \) on a finite-dimensional Hermitian space is called *non-negative* if it is self-adjoint and all its eigenvalues are positive or zero. Show, using the spectral theorem, that a non-negative linear map has an *unique* non-negative square root.