Mathematics V1208y
Honors Mathematics B

Assignment #5
Due February 26, 2016

Reading: Apostol §§5.1–5.10 (pp. 114–124) and §5.19 (pp. 138–141).

*1. For each of the matrices below, either diagonalize it (i.e. express it as $BDB^{-1}$ with $D$ diagonal) or prove that this is impossible. Hint: it helps to choose cleverly when expanding by minors.

\[
\begin{pmatrix}
20 & -9 \\
30 & -13
\end{pmatrix};
\begin{pmatrix}
8 & 4 \\
-9 & -4
\end{pmatrix};
\begin{pmatrix}
-1 & 4 & 4 \\
0 & -5 & -4 \\
0 & 8 & 7
\end{pmatrix}.
\]

2. Apostol §4.4 (p. 101) 1, 2, 3, *4, 6, 10, 11, 12.

3. Apostol §4.8 (pp. 107–8) 1, 2, 7, *11.

4. Apostol §4.10 (pp. 112–13) 2, 4, 6, *7, *8abc, 8d.

*5. Let $V$ be an $n$-dimensional vector space and let $T : V \to V$ be a linear map. Define the characteristic polynomial $\chi_T : \mathbb{R} \to \mathbb{R}$ by $\chi_T(\lambda) = \det(\lambda I - A)$, where $A$ is the matrix representing $T$ in any basis.

(a) Prove that $\chi_T$ does not depend on the choice of basis.

(b) Prove that $\chi_T$ is a polynomial function of $\lambda$ of degree $n$ with leading term $\lambda^n$.

*6. If $A$ is an upper-triangular square matrix, show that the eigenvalues of $A$ are exactly its diagonal entries $a_{ii}$. (“Exactly” means a number is an eigenvalue if and only if it’s one of the diagonal entries.)

7. Let $P_k$ be the space of polynomials of degree $\leq k$ as usual, and consider the maps $T : P_k \to P_k$ given by $T(p) = q$, where

(a) $q(x) = p'(x)$;
(b) $q(x) = xp'(x)$;
(c) $q(x) = p''(x) + p'(x) + p(x)$;
(d) $q(x) = p(x + 1)$.

In each case, find the eigenvalues, eigenfunctions (that is, functions that are eigenvectors) and determinant of $T$. In which cases is it diagonalizable?