From now on, all assignments are from Volume II of Apostol.

Reading: Apostol §4.

1. Apostol §3.6 (pp. 79–81) 1, 2, 3.

2. Apostol §3.11 (pp. 85–86) 1, 2, 3, 4, 7.

*3. Show that if $A$ is any $n \times n$ matrix, then $\det(cA) = c^n \det A$.

*4. For a square matrix $A$ with transpose $A^t$, prove that $\det A = \det A^t$ by showing that the right-hand side satisfies the axioms of the determinant. (Don’t follow the proof given in Apostol.)

*5. A square matrix $A$ is said to be upper-triangular if $A_{ij} = 0$ whenever $i > j$. Prove that if $A$ is upper-triangular and invertible, then its inverse is upper-triangular. Give a $2 \times 2$ example.

*6. Use expansion by minors to prove that the determinant of an upper-triangular matrix is the product of its diagonal entries. (Don’t follow the proof given in Apostol.)

*7. (a) For any square matrix $A$, show that $A + A^t$ is symmetric and $A - A^t$ is skew-symmetric.

(b) Use (a) to show that any square matrix can be expressed uniquely as a sum of one symmetric and one skew-symmetric matrix.

(c) Use a previous problem to show that if $n$ is odd, then any $n \times n$ skew-symmetric matrix has determinant 0.

(d) Show that any nonzero $3 \times 3$ skew-symmetric matrix has rank exactly 2.

Hint: note that diagonal entries must vanish, and rule out all other possibilities.

*8. Given fixed column vectors $Y_1, Y_2, \ldots, Y_{n-1} \in \mathbb{R}^n$, define a linear map $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(X) = \det(Y_1, Y_2, \ldots, Y_{n-1}, X),$$

that is, the determinant of the square matrix with columns $Y_1$ through $Y_{n-1}$ and $X$. Describe the kernel of $f$ in terms of the vectors $Y_1, \ldots, Y_{n-1}$, and prove that your description is correct.
*9. For $x_1, \ldots, x_n \in \mathbb{R}$ the Vandermonde determinant is defined to be

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$ 

Prove that it equals $\prod_{1 \leq i < j \leq n} (x_j - x_i)$, the product of all differences $x_j - x_i$ with $i < j$.

[Cf. problem 3a on p. 80 of Apostol.] Hint: subtract $x_1$ times the $k$th row from the $k+1$st for $k = n-1, n-2, n-3, \ldots, 1$. Then expand by minors and use induction.