Perturbative Analysis of Volatility Smiles

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Main Risks in Options Markets

Volatility changes in time

Index volatility is mean-reverting

It is negatively correlated with the price

A jump in price often entails a volatility jump

Most models ignore at least one of these risks
What Is the Shape of Smile?

S&P500 volatility surface on January 11, 1996

- Implied volatility decreases with strike price
- The skew slope is the greatest for short maturities

What underlying processes produce such skews?
Is the Skew Due to Jumps?

- **Jump Diffusion model**
  - between jumps \( \frac{dS_t}{S_t} = \mu^* dt + \sigma \, d\zeta(t) \)
  - in a jump \( S_t \rightarrow S_t e^{(\gamma + \delta \varepsilon)} \) \( \varepsilon \sim N(0,1) \)
  - jumps arrive with rate \( \lambda \)

\[
p_n = \frac{(\lambda T)^n \varepsilon^{-\lambda T}}{n!}
\]

- For S&P 500
  \[
  \begin{align*}
  \lambda &\sim 0.5 \\
  \gamma &\sim -0.2 \\
  \delta &\sim 0.05 \div 0.15
  \end{align*}
  \]

The jump diffusion model works well for short maturities.
What Happens at Longer Maturities?

- **Stochastic volatility**
  \[
  \begin{align*}
  dS_t / S_t &= \mu dt + \sqrt{v_t} \, dz_s(t) \\
  dv_t &= \kappa (\theta - v_t) + \sigma \sqrt{v_t} \, dz_v(t) \\
  Corr(dz_s,dz_v) &= \rho < 0
  \end{align*}
  \]

- **For S&P 500**
  \[
  \begin{align*}
  \sigma &\sim 0.8 \quad \text{(creates curvature)} \\
  \rho &\sim -0.7 \quad \text{(creates skew)} \\
  \kappa &\sim 1 \div 3
  \end{align*}
  \]

Stochastic volatility models work well for long maturities.
How to Combine Stochastic Volatility and Jump Diffusion?

- between jumps
  \[
  \begin{align*}
  \frac{dS}{S} &= \mu dt + \sqrt{v} dz_1 \\
  dv &= \kappa (\theta - v) dt + \sigma \sqrt{v} dz_2
  \end{align*}
  \]
  \(\text{Corr}(dz_1, dz_2) = \rho\)

- market crashes form a Poisson process with rate \(\lambda\)
  \[
  \begin{align*}
  \log S &\rightarrow \log S + \gamma_s + \delta_s \varepsilon \\
  \nu &\rightarrow \nu + \gamma_v
  \end{align*}
  \]

- the option price obeys the equation
  \[
  \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \kappa (\theta - v) \frac{\partial f}{\partial v} + \frac{1}{2} \nu \left\{ S^2 \frac{\partial^2 f}{\partial S^2} + \sigma^2 \frac{\partial^2 f}{\partial v^2} + 2 \rho \sigma S \frac{\partial^2 f}{\partial S \partial v} \right\} \\
  + \lambda E^* \left[f (Se^{\gamma_s + \delta_s \varepsilon}, \nu + \gamma_v) - f (S, \nu)\right] &= rf
  \]

European option prices can be computed analytically.
What Is the Distribution of Stock Prices?

- Call prices equal
  \[ C = S P_1 - K e^{-rT} P_0 \]

- Find the characteristic functional
  \[ f(t, \phi) = E^* \left[ e^{i \phi \ln(S/F)} \right] = \text{Fourier Transform of } P'_0 \]

- Use the affine ansatz
  \[ \hat{P}_n = e^{C(T-t, \phi) + D(T-t, \phi) \nu} \]
  to derive

\[
\begin{align*}
C(\tau, \varphi) &= C_H(\tau, \varphi) + \lambda \tau \left[ e^{i \varphi \gamma_s - \varphi^2 \delta_s^2 / 2} I(\tau) - 1 \right] \\
D(\tau, \varphi) &= D_H(\tau, \varphi) \\
I(\tau) &= \frac{1}{\tau} \int_0^\tau e^{\gamma_s D(t, \varphi)} dt = -\frac{2 \gamma_s}{\sigma^2} \int_0^\tau \frac{e^{-z} dz}{(1 + z / p_+)(1 + z / p_-)}
\end{align*}
\]

This model accounts for the main risks of options markets.
Does the Model Fit the Smile?

S&P500 volatility surface on June 11, 1997

The whole volatility surface is described by one set of constant parameters
Are Smile Parameters Stable Over Time?

- Volatility parameters:
  - current volatility $\sqrt{v}$
  - correlation $\rho$
  - vol of vol $\sigma$
  - long run volatility $\sqrt{\theta}$
  - mean reversion rate $\kappa$

- Market crash parameters:
  - crash rate $\lambda$
  - crash magnitude $\gamma_S$
  - vol jump magnitude $\gamma$

Mean reversion, correlation and crash size are constant
Patterns in Stochastic Volatility Parameters

Long run diffusion volatility is relatively stable.
What Is the Intuition?

- How does each source of risk affect the smile slope
  - at long maturities
  - at short maturities
- What is its effect on
  - ATM volatility
  - smile curvature
- For many models, the “weak smile expansion” is a good guide.
- However, the natural expansion is for the characteristic functional, not the implied volatilities.

How to construct the weak smile expansion?
Linking Characteristic Functionals to Implied Volatilities

- The characteristic functional

\[ F_t(\eta) = \int_0^\infty p(K) e^{i\eta \ln(K/F)} dK \]

The probability distribution

\[ p(K) = e^{rT} \frac{\partial^2 C}{\partial K^2} \]

- Introduce implied standard deviation

\( \phi = \sigma(K,T)\sqrt{T} \)

Parametrize \( \phi = \phi(z) \)

where \( z \equiv d_2 = \frac{\ln(F/K) - \phi/2}{\phi} = \ln(M/K), \quad M = Fe^{-\phi^2/2} \)

- Then

\[ p(K) dK = N'(z) \ d\phi \left\{ -1 + \frac{z}{\phi / \phi + z + \phi} - \frac{\partial}{\partial z} \left( \frac{1}{\phi / \phi + z + \phi} \right) \right\} \]

\( \phi \equiv \frac{\partial \phi}{\partial z} \)
Linking Characteristic Functionals to Implied Volatilities

• Changing the integration variable to \( z \) and integrating by parts

\[
F(\eta) = \int_{-\infty}^{+\infty} dz \ N'(z) e^{-i\eta \phi \left( \frac{1}{2} \phi^2 + z \right)} (1 + i\eta \phi)
\]

• In terms of \( w \equiv z + i\eta \phi(z) \)

\[
F(\eta) = \int_{-\infty}^{+\infty} dw \ N'(w) e^{-\frac{1}{2} \eta(\eta+i)\phi^2(w)}
\]

\[
w = \frac{\ln(F / K)}{\phi(w)} + \left( i\eta - \frac{1}{2} \right) \phi(w)
\]

\( F(\eta) \) is related to the analytic continuation of \( \phi \)
What Is the First Order Perturbation?

- Assume \( \varphi^2(w) \equiv \varphi_0^2 + \psi_1(\eta, w) \)
  with \( \varphi_0 \) independent of \( w \).
- Then \( F(\eta) = e^{-\frac{1}{2} \varphi_0^2 \eta (\eta + i)} \{1 + F_1(\eta)\} \)

\[
\int_{-\infty}^{+\infty} dw N'(w) \psi_1(\eta, w) = -\frac{2}{\eta(\eta + i)} F_1(\eta)
\]

\[
w = -\frac{x}{\varphi_0} + \varphi_0 \left( i\eta - \frac{1}{2} \right) + O(\psi_1)
\]
- As a result

\[
\psi_1(x) = -\frac{2\varphi_0}{\sqrt{2\pi}} e^{\frac{x^2}{2\varphi_0^2}} \int_{-\infty}^{+\infty} F_1(\eta^*) d\eta^* e^{-\frac{1}{2} \varphi_0^2 \eta^{*2} - ix\eta^*} \quad \eta^* = \eta + i / 2
\]

The smile slope is a simple integral of \( F_1(\eta^*) \)
What Is the Effect of Price Jumps?

- In the Merton model
  - **The ATM smile slope**
    \[ \left. \frac{\partial \psi_1}{\partial x} \right|_{x=0} = 2\lambda T (e^\gamma - 1) \quad \Rightarrow \quad \left. \frac{d\sigma}{dx} \right|_{x=0} = \frac{\lambda (e^\gamma - 1)}{\sigma_0} \]
    - stable calibration of expected loss \( \lambda (e^\gamma - 1) \)
    - more noise in \( \lambda \) and \( \gamma \)
  - **The smile curvature** (\( \gamma = 0 \))
    \[ \left. \frac{\partial^2 \psi_1}{\partial x^2} \right|_{x=0} = \frac{\lambda T \delta^4}{4\phi_0^2} \quad \bar{\sigma} \approx \sigma_0 + \frac{\lambda \delta^4 x^2}{16 \sigma_0^3 T} \]
    - at small \( \delta \), very straight skews
    - strong dependence on \( \delta \) when \( \delta \) is large
What Is the Effect of Stochastic Volatility?

- **Black-Scholes variance**  
  \[ \varphi_0^2 \equiv \theta T + \frac{\nu - \theta}{\kappa} (1 - e^{-\kappa T}) \]

- **As** \( T \to 0 \), \( \psi_1'(x) = \frac{1}{2} \rho \sigma T \)
  Hence the smile slope (in stdev space)
  \[ \frac{1}{\sigma_0} \frac{d\sigma}{dz} = \sqrt{T} \frac{d\sigma}{dx} = \frac{\rho \sigma}{4\sqrt{v_0}} \sqrt{T} \propto 0.16 \]

- **As** \( T \to \infty \), \( \psi_1'(x) = \frac{\rho \sigma}{\kappa} \)
  \[ \frac{1}{\sigma_0} \frac{d\sigma}{dz} = \frac{\rho \sigma}{2\kappa \sqrt{\theta T}} \propto 0.06 \]
  - calibration of \( \rho \sigma \) more stable
  - long run skew often too flat

The long run Heston smile is often too flat
How Jumps in Volatility Change the Picture?

- If $\gamma = \delta = 0$, only the change in volatility level

$$\psi_1(x) = -(\lambda T)(\gamma, T) \frac{1 - \kappa T - e^{-\kappa T}}{(\kappa T)^2} \to \frac{1}{2} (\lambda T)(\gamma, T) \quad \text{as} \quad T \to 0$$

$$\to \frac{\lambda \gamma_v}{\kappa} T \quad \text{as} \quad T \to \infty$$

- Interaction of vol jumps with price jumps

$$\psi_1(x) = \frac{(\lambda T)(\gamma_v, T)}{2} \frac{\gamma}{\varphi_0^2} \frac{1 - \kappa T - e^{-\kappa T}}{(\kappa T)^2}$$

$$\text{as} \quad T \to 0, \quad \bar{\sigma}_x' = \frac{\lambda \gamma}{\sigma_0} \to (1 + \alpha_0) \frac{\lambda \gamma}{\sigma_0} \quad \alpha_0 = \frac{\gamma_v}{4\sigma_0^2}$$

for the jump from 15% to 35%

$$\gamma_v \propto 0.10 \Rightarrow \alpha_0 \propto 1.0$$

$$\text{as} \quad T \to \infty, \quad \psi_1(x) = \frac{\lambda \gamma \gamma_v}{\kappa \theta} \quad \text{vs.} \quad \frac{\rho \sigma}{\kappa} \Rightarrow \alpha_\infty = \frac{\lambda \gamma \gamma_v}{\rho \sigma \theta} \propto 0.9$$

Volatility jumps significantly affect the skew
What Is the Delta?

- When the spot moves, the smile can move too

Thus \[ \Delta = \frac{\delta C}{\delta S} = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial \sigma} \frac{\delta \sigma}{\delta S} \]

- Three regimes

Black-Scholes smile stays fixed

Relative Smile smile floats with spot

Fixed Implied Tree \( \sigma(S, t) \) unchanged

Stochastic volatility and jump diffusion yield relative smiles
How to Minimize the P(L) Variance?

- Given $\delta S$ and $\delta \nu$, 

  $$\delta C = \Delta \delta S + \Lambda \delta \nu$$

  Hedge with $y$ shares: 

  $$P / L = \delta C - y \delta S$$

- In a stochastic volatility model

  $$\text{Var}(P / L) = (\Delta - y)^2 \text{Var}(\delta S) + 2(\Delta - y)\Lambda \text{Cov}(\delta S, \delta \nu) + \Lambda^2 \text{Var}(\delta \nu)$$

  Minimize with respect to $y$: 

  $$y = \Delta + \rho \sigma \Lambda / S < \Delta$$

- Since 

  $$\Delta = \Delta_{BS} - \Lambda \bar{\sigma}^2 \left( x \right) / S$$

  $$y = \Delta_{BS} + \rho \sigma \Lambda / 2S$$

  Optimal “risk management” delta $< \Delta_{BS}$
What Is the Meaning of the Implied Tree?

- Imagine the world is described by a stochastic volatility model, but we hedge with the implied tree model.

- Then the smile slope
  \[
  \frac{d\bar{\sigma}}{dx} = \frac{\rho\sigma}{4\bar{\sigma}}
  \]

- When we move the spot, keeping the implied tree fixed,
  \[
  \frac{\delta\bar{\sigma}^2(x,T)}{\delta x} = \frac{1}{T} \int_0^T dt \ E_{BB} \left[ \frac{\partial\bar{\sigma}^2(\xi,t)}{\partial \xi} \right]_{\xi = \xi_{BB}(t)}
  \]

  Thus
  \[
  \Delta_{IT} = \Delta_{BS} + \rho\sigma\Lambda / 2S = y
  \]

Implied tree delta mimicks the risk management delta.
Summary and Overview

- Stochastic volatility and market jumps produce a skewed surface of implied volatilities
- The effect of volatility jumps on the skew is highly significant
- Perturbative expansions are a useful tool for understanding the smile
- The optimal delta depends on the dynamics of volatility