Modelling Volatility and Volatility Derivatives

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What Determines the Smile Term Structure?

S&P500 volatility surface on January 11, 1996

Market crashes drive the short-term smile
Uncertainty in volatility drives the long-term smile
How to Combine Stochastic Volatility and Jump Diffusion?

- **between jumps**
  \[
  \begin{align*}
  \frac{dS}{S} &= \mu dt + \sqrt{v} dz_1 \\
  dv &= \kappa(\theta - v)dt + \sigma \sqrt{v} dz_2 \\
  \text{Corr}(dz_1, dz_2) &= \rho
  \end{align*}
  \]

- **market crashes form a Poisson process with rate** \( \lambda \)
  \[
  \begin{align*}
  \log S &\rightarrow \log S + \gamma_s + \delta_s \varepsilon \\
  \varepsilon &\sim N(0,1)
  \end{align*}
  \]

- **the option price obeys the equation**
  \[
  \begin{align*}
  \frac{\partial f}{\partial t} + \mu^* S \frac{\partial f}{\partial S} + \kappa(\theta - v) \frac{\partial f}{\partial v} + \frac{1}{2} v \left\{ S^2 \frac{\partial^2 f}{\partial S^2} + \sigma^2 \frac{\partial^2 f}{\partial v^2} + 2 \rho \sigma S \frac{\partial^2 f}{\partial S \partial v} \right\} \\
  + \lambda E^* \left[ f(S e^{\gamma_s + \delta_s \varepsilon}, v + \gamma_v) - f(S, v) \right] &= rf
  \end{align*}
  \]

European option prices can be computed analytically.
What are the European Option Prices?

- Call prices equal
  \[ C = S P_1 - K e^{-rT} P_0 \]

- The Fourier Transforms of \( P_1 \) and \( P_0 \) have the affine form
  \[ \hat{P}_n = e^{C(T-t,\varphi)+D(T-t,\varphi)v} \]

- \( C(\tau, \varphi) \) and \( D(\tau, \varphi) \) obey the first order equations in \( \tau \)
  \[
  \begin{aligned}
  C(\tau, \varphi) &= C_H(\tau, \varphi) + \lambda \tau \left[ e^{i \varphi \delta^2 \tau^2 / 2} I(\tau) - 1 \right] \\
  D(\tau, \varphi) &= D_H(\tau, \varphi)
  \end{aligned}
  \]

  \[
  I(\tau) = \frac{1}{\tau} \int_0^\tau e^{\gamma v D(t,\varphi)} dt = -\frac{2\gamma v}{p_+ p_-} \int_0^{\gamma v D(\tau,\varphi)} \frac{e^{-z} dz}{(1+z/p_+)(1+z/p_-)}
  \]

  Computed efficiently using the Fast Fourier Transform

  There is no need to sum over jumps
Does the Model Fit the Smile?

S&P500 volatility surface on June 11, 1997

The whole volatility surface is described by one set of constant parameters
Does the Model Fit the Smile?

S&P500 volatility surface in August, 1999

The whole volatility surface is described by one set of constant parameters
Are Smile Parameters Stable Over Time?

● Volatility parameters:
  - current volatility $\sqrt{v}$
  - correlation $\rho$
  - vol of vol $\sigma$
  - long run volatility $\sqrt{\theta}$
  - mean reversion rate $\kappa$

● Market crash parameters:
  - crash rate $\lambda$
  - crash magnitude $\gamma_s$
  - vol jump magnitude $\gamma$

Mean reversion, correlation and crash size are constant
Patterns in Stochastic Volatility Parameters

Long run diffusion volatility is relatively stable
Are Exotics Prices Different?

Down-and-out call maturity 3 years

Deltas in the two models

<table>
<thead>
<tr>
<th>strike</th>
<th>barrier level</th>
<th>Stochastic Volatility / Jump Diffusion</th>
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<tbody>
<tr>
<td>90</td>
<td>95</td>
<td>1.46 1.13 0.98 0.86</td>
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<th>barrier level</th>
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<td>1.11 1.24 1.02 0.94</td>
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<tr>
<td>120</td>
<td>90</td>
<td>0.71 0.77 0.64 0.61</td>
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</table>

Deterministic volatility models may misprice barrier options
What are the Risk Premia?

- The risk-neutral drift of variance and the jump rate contain risk premia: \( \sigma_m^* = \sqrt{\theta^*} > \sigma_m = \sqrt{\theta} \quad \lambda^* > \lambda \)

- In the power utility model with \( U(C) \propto C^\alpha \)
  
  \[
  dC_t / C_t = \mu_c dt + \sigma_c \sqrt{v_t} dz_c + (e^{\gamma_c} - 1) dq
  \]
  
  - crash rate \( \lambda^* = \lambda e^{(\alpha-1)\gamma_c} \)
  
  - mean reversion \( \kappa^* = \kappa - (\alpha-1)\sigma_c \sigma \rho_{cv} \)
  
  - total return \( R = r - (\alpha-1)\nu_t \sigma_c \sigma_x \rho_{cx} + (e^{\gamma_x} - 1)(\lambda - \lambda^*) \)

- Rough estimates for S&P500: \( \lambda \propto 0.1, \quad \lambda^* \propto 0.5 \)
  \( \gamma_x \propto -0.1, \quad \alpha-1 \propto -16, \quad \gamma_c \propto -0.1 \quad \Rightarrow \quad \kappa^* - \kappa \propto -1.5, \quad \lambda^* \propto 5\lambda \)

Risk premia are roughly consistent with historic observations
Derivatives on Realized Volatility

- Volatility swap:
  \[ \text{payout at maturity} = \text{notional} \times \left[ \sigma_{\text{historic}} - \sigma_{\text{agreed vol}} \right] \]

- Variance swap:
  \[ \text{payout at maturity} = \text{notional} \times \left[ \sigma_{\text{historic}}^2 - \sigma_{\text{agreed var}}^2 \right] \]

- \( \sigma_{\text{historic}} \) is the standard deviation of realized returns
  \[ \sigma_{\text{historic}} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (r_i - \bar{r})^2}, \quad r_i = \ln \left( \frac{S_{i+1}}{S_i} \right) \]

- To hedge a variance swap, buy 2 log contracts \( f(S_T) = \log(S_T / F) \)
  \[ \Delta t \left[ \frac{1}{2} \sigma_{\text{realized}}^2 S^2 \Gamma_f - \frac{1}{2} \sigma_{\text{implied}}^2 S^2 \Gamma_f \right] = \frac{1}{2} \Delta t \left( \sigma_{\text{implied}}^2 - \sigma_{\text{realized}}^2 \right) \]

Without jumps, the log hedge is model-independent
How to Find Expected Volatility?

- Evaluate the characteristic functional
  \[ f(v, t \mid z) = E_t^* \left[ e^{-zV_t} \mid F_t \right] \]
  \[ V_t = \int_0^T v(\tau)d\tau + \sum_{n=1}^{N_j} (\gamma_s + \delta_s \epsilon)^2 \]

- Expected variance:
  \[ E\{V_t\} = -\left( \frac{\partial f}{\partial z} \right) \bigg|_{z=0} \]

- Expected volatility:
  \[ E\left\{ \sqrt{V_t} \right\} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E(e^{-zV_t})}{z^{3/2}}dz \]

- Expected payout of a variance call option:
  \[ E\{\max(V_t - K, 0)\} = \frac{1}{\pi} \text{Re} \int_0^\infty \frac{1 - E(e^{i\varphi(V_t - K)})}{\varphi^2} d\varphi + \frac{1}{2} \left[ E\{V_t\} - K \right] \]

The characteristic functional is easily found in closed form
What are the Resulting Prices?

- Characteristic functional has the form \( f(v, t \mid z) = e^{C(t, z)+D(t, z)v} \)

- The final result is \( f(v, t \mid z) = \psi_{sv}(z) \psi_j(z) \)

\[
\psi_{sv}(z) = e^{\zeta \tau/2} \left[ \cosh\left(\frac{\mu \tau}{2}\right) + \frac{1}{\mu} \sinh\left(\frac{\mu \tau}{2}\right) \right]^{-\zeta} \exp\left\{-\frac{2zv}{\kappa[1+\mu \coth(\mu \tau/2)]}\right\}
\]

\[
\psi_j(k) = \exp\left\{ -\lambda T \left( 1 - \frac{1}{\sqrt{1+2k\delta^2}} e^{-k\delta^2/2} \left( \frac{T}{T_0} \int e^{D(t)\gamma} \, dt \right) \right) \right\}
\]

\[
D(t \mid z) = -\frac{2z}{\kappa} \frac{1}{1+\mu \coth(\mu \tau/2)}
\]

- What is the difference between expected vol and variance?

<table>
<thead>
<tr>
<th></th>
<th>Swap</th>
<th></th>
<th>Spread to implied vol</th>
<th>Convexity adjustment</th>
</tr>
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<tbody>
<tr>
<td>volatility</td>
<td>variance</td>
<td>implied vol</td>
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</tr>
<tr>
<td>combined model</td>
<td>26.34%</td>
<td>31.20%</td>
<td>26.53%</td>
<td>-0.19%</td>
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<tr>
<td>stochastic vol</td>
<td>28.64%</td>
<td>31.29%</td>
<td>26.53%</td>
<td>2.11%</td>
</tr>
<tr>
<td>jump diffusion</td>
<td>27.55%</td>
<td>30.06%</td>
<td>26.53%</td>
<td>1.02%</td>
</tr>
</tbody>
</table>

The convexity adjustment due to jumps may be substantial

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How to Hedge Against Jumps?

- Constant diffusion volatility $\Rightarrow$ no risk between crashes
- An exact hedge against a crash is $x = \gamma^2 / (e^\gamma - 1 - \gamma)$ of the log security together with its delta $\Delta = (1/S) e^{-r(T-t)}$
- During a crash, gain $\left[ \log(S_{i+1} / S_i) \right]^2 = \gamma^2$ on the variance contract which offsets exactly by the loss on the hedge:
  $$x \left\{ \log(S_t e^\gamma) - \log(S_t) \right\} + \Delta (S_t e^\gamma - S_t) = -x(e^\gamma - 1 - \gamma)$$
- For S&P500, with jump size $\gamma \approx -0.15$ the hedge ratio $x \approx 2(1-\gamma / 3) \approx 2.10$

In the stochastic volatility & jump diffusion model the optimal hedge ratio is closer to 2.0
What is the Optimal Log Hedge?

- Hedge with $x$ log contracts and $y$ shares
- Find $x$ and $y$ to minimize the expected P/L variance

\[
\lambda \left\{ \Delta C + x \Delta L + y \Delta S \right\}^2 + \nu_t \left\{ \sigma^2 (\Lambda_C + x \Lambda_L)^2 + y^2 + 2 \rho \sigma y (\Lambda_C + x \Lambda_L) \right\}
\]

- The optimal amounts and exposures change over time

\[
x/(1/\sigma_{\text{exp}})
\]

![Volatility Swap](image)

![Variance Swap](image)
Summary and Overview

- Model matches the whole smile with one set of parameters
- It provides a realistic representation of risks
- The impact of jumps on the pricing and hedging of volatility derivatives is significant