COMPLEX SURFACE SINGULARITIES WITH INTEGRAL HOMOLOGY SPHERE LINKS

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Abstract. The Casson Invariant Conjecture (CIC) asserts that for a complete intersection surface singularity whose link is an integral homology sphere, the Casson invariant of that link is one-eighth the signature of the Milnor fiber. We study a large class of such complete intersections, those of “splice type,” and boldly conjecture that all Gorenstein singularities with homology sphere links are equisingular deformations of singularities of this type. We propose, and verify in a non-trivial case, a stronger conjecture than the CIC for splice type complete intersections: a precise topological description of the Milnor fiber. We prove the CIC itself for a special class of splice type singularities; this result includes all previously proven cases of the Conjecture. We show also that a singularity with homology sphere link is of splice type (up to equisingular deformation) if and only if some naturally occurring knots in the singularity link are themselves links of hypersurface sections of the singular point.

In [22] we formulated the Casson Invariant Conjecture. Let \((X, o)\) be an isolated complete intersection surface singularity whose link \(\Sigma\) is an integral homology 3-sphere. Then the Casson invariant \(\lambda(\Sigma)\) is one-eighth the signature of the Milnor fiber of \(X\).

The interest of this conjecture is its suggestion that the Milnor fiber is a “natural” 4-manifold which is attached to its boundary \(\Sigma\), and for which the signature computes the Casson invariant exactly (and not just mod 2). Specifically, it implies that for a complete intersection singularity whose link is a homology sphere, analytic invariants like the Milnor number and geometric genus are determined by the link. (Such results are known to be false for general hypersurface singularities.)

In [22] we verified the Casson Invariant Conjecture for Brieskorn-Pham complete intersections by direct computation. It was a challenge to find other examples; having done so, the conjecture was verified in these cases, with the serious work being calculation of the signature.

The topological types of normal surface singularities with homology sphere link may be classified, and form a rich class ([7]); they are obtained by repeatedly splicing together the links \(\Sigma(p_1, \cdots, p_n)\) of Brieskorn-Pham complete intersections along naturally occurring knots. But it is completely unclear which topological types may be realized by complete intersection (or even Gorenstein) singularities.

In a parallel paper [25], we describe how “most” homology sphere singularity links do, in fact, arise as links of complete intersection singularities, and we give explicit equations. These equations, which we call “splice type,” generalize the

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Brieskorn-Pham complete intersections which correspond to “one-node” splice diagrams). One may think in terms of an operation of splicing the defining equations of two singularities which corresponds to splicing the links. While it is easy to see the effect of splicing on the Casson invariants of the links, it is extremely hard to calculate the signature of the Milnor fiber for the new singularity (let alone understand its topology).

We conjecture a topological construction that, when splicing two singularities, creates the new Milnor fiber out of the old ones, extending the operation of splicing on the boundaries. This conjecture easily implies (and hence motivates) the Casson Invariant Conjecture for such singularities (Corollary 3.2 below). We succeed in proving it in a non-trivial case:

**Theorem 1.** For a singularity \( z^n + g(x, y) = 0 \) with homology sphere link, the Milnor fiber is formed by the conjectured topological construction.

Though the Casson Invariant Conjecture for this case follows, it had already been proven in [22] (by a much less conceptual proof), and recently by Collin and Saveliev [4] using equivariant Casson invariants.

As for the signature itself, the following result includes all previously proved cases of the Casson Invariant Conjecture, including the above

**Theorem 2.** The Casson Invariant Conjecture is true for complete intersection singularities of splice type for which the nodes of the splice diagram are in a line.

We prove this theorem by computing geometric genus \( p_g \) rather than signature, since \( p_g \) has a better chance of being computed directly from defining equations. We pointed out in [22] that the Casson Invariant Conjecture may be formulated in terms of geometric genus.

**Casson Invariant Conjecture** (Version 2). Let \((X, o)\) be a Gorenstein surface singularity with integral homology 3-sphere link \( \Sigma \). Then the Casson invariant \( \lambda(\Sigma) \) equals \( -p_g(X, o) - \frac{1}{8} C(\Sigma) \), where \( C(\Sigma) \) is the characteristic number \( c_1^2 + c_2 - 1 \) of any good resolution of \( X \) (this is a topological invariant).

This version implies the previous version by formulas of Laufer and Durfee (see proof of Theorem 3.3), but it is \textit{a priori} more general since \((X, o)\) is not required to be a complete intersection. However, having introduced splice type singularities, we conjecture (see Conjecture 2 in Section 2 for a precise version):

**Splice Type Conjecture.** Any Gorenstein surface singularity with integral homology sphere link is a complete intersection, and is of splice type up to a natural notion of equisingular deformation.

Implicit in this conjecture is a new and unexpected necessary condition (the "semigroup condition") on a splice diagram (and hence on a resolution diagram) in order that it come from a Gorenstein singularity. The conjecture implies that the topology of a homology sphere link determines a Gorenstein singularity uniquely up to equisingularity—a kind of “tautness.” (One may compare and contrast with the equations of plane curve singularities with given Puiseux pairs.) An elementary case for which we can show the conjecture is for any singularity \( z^n + g(x, y) = 0 \) with homology sphere link (Corollary 8.2).

To understand the “semigroup condition,” recall that a homology sphere link \( \Sigma \) of a normal surface singularity \((X, o)\) has a number of natural knots, one for each leaf of the splice diagram (or equivalently, of the resolution graph). For a splice type...
singularity these knots are cut out by coordinate hyperplanes. We prove, conversely (see Theorem 8.1 for a more precise version):

**Theorem 3.** For a normal surface singularity \((X, o)\) with homology sphere link, if all the knots associated to leaves of the splice diagram are links of hypersurface sections of \(X\), then the semigroup condition is fulfilled, and \(X\) is a complete intersection of splice type up to equisingular deformation.

So, one might expect the Casson Invariant Conjecture to follow from its truth for complete intersections of splice type. But we expect things to go in the opposite direction: a proof of the Casson Invariant Conjecture (perhaps symplectic or gauge-theoretic) might allow one to deduce the form of defining equations. This happens for instance in the one-node case: we proved in [22] that a Gorenstein singularity \((X, o)\) with link \(\Sigma(p_1, \ldots, p_n)\) is an equisingular deformation of the corresponding Brieskorn-Pham complete intersection if and only if the Casson Invariant Conjecture holds for \(X\) (equivalently, \(X\) has the same geometric genus as the Brieskorn-Pham complete intersection). We remark that A. Némethi [18] has proved these equivalent statements hold for weakly elliptic singularities, e.g., for \(\Sigma(2, 3, 6k + 5)\).

In [23, 24] we proposed a more general version of the Splice Type Conjecture: Any \(\mathbb{Q}\)-Gorenstein surface singularity with \(\mathbb{Q}\)-homology sphere link has as universal abelian cover an equisingular deformation of a complete intersection singularity of splice type (using a generalized notion of splice diagram). In [25] we discuss this further and prove one direction, namely that the equations of splice type do indeed give the desired topology. In particular, the equations of splice type of the current paper give singularities with the expected homology sphere links, so we do not give a proof here.

On the other hand, the contents of this paper are somewhat transverse to [25], since in [24, 25] we offer no guess as to the topology or the signature of the Milnor fiber of the universal abelian cover. Though [22] wondered about a generalization involving the Casson-Walker invariant, computations for Seifert fibered rational homology spheres by Lescop [15, 16] showed the naive generalization fails (see also [5]). Lim’s result [14] suggests looking at a Seiberg-Witten invariant, and a recent generalization along these lines of the Casson Invariant Conjecture to \(\mathbb{Q}\)-Gorenstein \(\mathbb{Q}\)-homology spheres has been offered by Némethi and Nicolaescu [19].

We offer now a road map to help readers go through this paper.

Sections 1 and 2 are introductory. In Section 1, we review from [7] the definition of splice diagrams and the topological description of homology sphere links; further details are found in the Appendix (Section 9), where we also give an improved description of the relationship between splice diagrams and plumbing (or resolution) graphs. We also introduce the important “semigroup condition.” In Section 2 we associate “splice type equations” to any splice diagram with semigroup condition; this provides a wealth of examples of complete intersections with homology sphere links.

Sections 3 to 5 discuss the Milnor Fiber Conjecture. Section 3 introduces this conjecture, which describes the topology of the Milnor fiber of splice type singularities, and which would imply the Casson Invariant Conjecture. The discussion leads to Theorem 3.3, which describes under what condition the Casson Invariant Conjecture holds for a diagram obtained by splicing diagrams for which the conjecture is known. This involves the relationship between signature and geometric genus,
and the key is to understand the behavior of the topological invariant \( C(\Delta) \) of the link under splicing. This is done in Theorem 3.4, whose proof, using numerics of splice diagrams, takes up the following section (Section 4).

Section 5 verifies the Milnor Fiber Conjecture for equations of the form \( z^n = f(x,y) \), by careful topological construction of the Milnor fiber. This uses a description of plane curve singularities in terms of splice diagram equations.

The remaining sections address Theorems 2 and 3 and are for the most part independent of the preceding sections (although Theorem 3.3 is used in Section 7).

Section 6 develops some theory of semigroups and monomial curves that is needed in the next two sections. In particular, it includes a new characterization of complete intersection monomial curves in terms of one-dimensional analogues of splice type singularities (Theorem 6.1 and its scholium).

Section 7 has as its goal the inductive calculation of the geometric genus \( p_g \) for a splice type singularity. Every node \( v \) of the splice diagram gives a valuation (or weight function) \( \nu \) of the singularity; a key result (Theorem 7.3) states that the associated graded ring associated to \( \nu \) is an integral domain, whose normalization is a Brieskorn-Pham complete intersection. Though \( p_g \) is the colength of a fairly explicit ideal determined by all \( \nu \)'s, only when all the nodes of the splice diagram are on a line can we find “good bases” allowing calculation of this number. The main result is Theorem 7.6.

Finally, Section 8 examines the key property of a splice type singularity: the natural knots in the link associated to leaves in the splice diagram are obtained by setting a coordinate equal to 0. We prove (Theorem 8.1) that conversely any normal surface singularity with homology sphere link, and for which the natural knots are hypersurface sections, is in fact an equisingular deformation of a splice type singularity. In fact, more sharply, it is a “higher weight deformation” — one obtained by adding to each equation only terms of higher \( \nu \)-weight, for the valuations corresponding to the nodes of the diagram.

As mentioned above, Section 9 is an appendix on splice diagrams.

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1. Splice diagrams for integral homology sphere links

For more details on splicing see the Appendix (Section 9).

Recall that a splice diagram is a finite tree with vertices only of valency 1 ("leaves") or \( \geq 3 \) ("nodes") and with a collection of integer weights at each node, associated to the edges departing the node. The following is an example.

For an edge connecting two nodes in a splice diagram the edge determinant is the product of the two weights on the edge minus the product of the weights adjacent to the edge. Thus, in the above example, the one edge connecting two nodes has edge determinant \( 77 - 60 = 17 \).
The splice diagrams that classify homology sphere singularity links satisfy the following conditions on their weights:

- the weights around a node are positive and pairwise coprime;
- the weight on an edge ending in a leaf is $> 1$;
- all edge determinants are positive.

More general splice diagrams appear for other situations (see, e.g., [7] and [24]), but we will only consider splice diagrams satisfying the above conditions here.

**Theorem 1.1 ([7]).** The homology spheres that are singularity links are in one-one correspondence with splice diagrams satisfying the above conditions.

The splice diagram and resolution diagram for the singularity determine each other uniquely, and describe how to construct the link by splicing or by plumbing. One method to compute the resolution diagram from the splice diagram is given in [7]. We describe an easier method in the appendix to this paper (Section 9), where we also recall the topological meaning of splicing and describe how to compute the splice diagram from the resolution diagram for a singularity.

The following notations will be used extensively in this paper.

**Notation.** For a node $v$ and an edge $e$ at $v$, let $d_{ve}$ be the weight on $e$ at $v$, and $d_v$ the product of the $d_{ve}$ over all such $e$. Let $\Delta_{ve}$ be the subgraph of $\Delta$ cut off from $v$ by $e$. For any pair of vertices $v$ and $w$, let $\ell_{vw}$ (the linking number) be the product of all the weights adjacent to, but not on, the shortest path from $v$ to $w$ in $\Delta$. We also consider $\ell'_{vw}$, the same product but excluding weights around $v$ and $w$. Thus if $v$ is a node and $w$ is a leaf in $\Delta_{ve}$, then

$$\ell_{vw} d_{ve} = \ell'_{vw} d_v.$$ 

**Definition 1.2 (Semigroup Condition).** Let $\Delta$ be a splice diagram. We say $\Delta$ satisfies the semigroup condition if, for each node $v$ and adjacent edge $e$, the edge-weight $d_{ve}$ is in the semigroup $\mathbb{N}\langle \ell'_{vw} : w \text{ a leaf of } \Delta \text{ in } \Delta_{ve} \rangle$.

Equivalently, the product $d_v$ of the edge-weights adjacent to $v$ is in the semigroup

$$\mathbb{N}\langle \ell_{vw} : w \text{ a leaf of } \Delta \text{ in } \Delta_{ve} \rangle.$$ 

For instance, in the two-node splice diagram above, let $v$ be the leftmost node and $w$ the upper right hand leaf. Then $\ell_{vw}$ equals $2 \cdot 3 \cdot 5$, while $\ell'_{vw} = 5$; the semigroup condition is satisfied at that node since 7 is in the semigroup generated by 2 and 5.

If a splice diagram satisfies the semigroup condition, we will write down complete intersection equations that give a singularity with the given link. We conjecture conversely that the semigroup condition is necessary for the link to be realized by a complete intersection — even a Gorenstein — singularity. The following is a special case of Conjecture 2 of [24].

**Conjecture 1 (Gorenstein implies Semigroup Condition).** If a surface singularity with homology sphere link is Gorenstein, then its splice diagram satisfies the semigroup condition.
For example, consider the splice diagram

\[ \Delta = \begin{tikzpicture}
  \node (a) at (0,0) [circle, draw] {}; 
  \node (b) at (1,0) [circle, draw] {}; 
  \node (c) at (2,0) [circle, draw] {}; 
  \node (d) at (3,0) [circle, draw] {}; 
  \node (e) at (4,0) [circle, draw] {}; 
  \draw (a) -- (b) node [midway, above] {\(p\)}; 
  \draw (b) -- (c) node [midway, above] {\(r\)}; 
  \draw (c) -- (d) node [midway, above] {\(p'\)}; 
  \draw (d) -- (e) node [midway, above] {\(q\)}; 
\end{tikzpicture} \]

with \(p, q, r \) and \( p', q', r' \) pairwise coprime triples of positive integers satisfying \( rr' > pqp'q' \). Then \( \Delta \) satisfies the semigroup condition if and only if

\[ r \in \mathbb{N}(p', q') \quad \text{and} \quad r' \in \mathbb{N}(p, q). \]

(Note \( r \) is automatically in the semigroup \( \mathbb{N}(p', q') \) if it is greater than or equal to the conductor \((p' - 1)(q' - 1)\).) In particular, the resolution diagram

\[ T = \begin{tikzpicture}
  \node (a) at (0,0) [circle, draw] {}; 
  \node (b) at (1,0) [circle, draw] {}; 
  \node (c) at (2,0) [circle, draw] {}; 
  \node (d) at (3,0) [circle, draw] {}; 
  \node (e) at (4,0) [circle, draw] {}; 
  \draw (a) -- (b) node [midway, above] {-2}; 
  \draw (b) -- (c) node [midway, above] {-7}; 
  \draw (c) -- (d) node [midway, above] {-3}; 
  \draw (d) -- (e) node [midway, above] {-1}; 
\end{tikzpicture} \]

gives the splice diagram

\[ \Delta = \begin{tikzpicture}
  \node (a) at (0,0) [circle, draw] {}; 
  \node (b) at (1,0) [circle, draw] {}; 
  \node (c) at (2,0) [circle, draw] {}; 
  \node (d) at (3,0) [circle, draw] {}; 
  \node (e) at (4,0) [circle, draw] {}; 
  \draw (a) -- (b) node [midway, above] {2}; 
  \draw (b) -- (c) node [midway, above] {1}; 
  \draw (c) -- (d) node [midway, above] {37}; 
  \draw (d) -- (e) node [midway, above] {2}; 
\end{tikzpicture} \]

which does not satisfy the semigroup condition, since 1 is not in the semigroup generated by 2 and 3. We would therefore expect that there is no Gorenstein singularity with this resolution.

2. Equations associated to a splice diagram

Let \( \Delta \) be a splice diagram satisfying the semigroup condition. We will write down a system of complete intersection equations that give a singularity with the corresponding link. Associate a variable \( z_w \) to each leaf \( w \) of the splice diagram. To each node \( v \) of the splice diagram, we will associate \((\delta_v - 2)\) equations, where \( \delta_v \) is the valency of the node. If \( n \) is the number of leaves, then it is easy to check that \( n - 2 = \sum (\delta_v - 2) \) (summed over the nodes of \( \Delta \)), so this will give the right number of equations.

Fix a node \( v \). For each leaf \( w \) we give the variable \( z_w \) weight \( \ell_{vw} \) (we call this the \( v \)-weight of \( z_w \)). For each edge \( e \) at \( v \) the semigroup condition lets us write

\[ d_v = \sum_w \alpha_{vw} \ell_{vw}, \quad \text{sum over the leaves } w \text{ of } \Delta \text{ in } \Delta_{ve}, \quad \text{with } \alpha_{vw} \in \mathbb{N}. \]

Equivalently,

\[ d_{ve} = \sum_w \alpha_{vw} \ell'_{vw} \quad \text{sum over the leaves } w \text{ of } \Delta \text{ in } \Delta_{ve}. \]

We define an admissible monomial (associated to the edge \( e \) at the node \( v \)) to be a monomial \( \prod_w z_{w}^{\alpha_{vw}} \), the product over leaves \( w \) in \( \Delta_{ve} \), with exponents satisfying the above equations. Thus an admissible monomial \( M_{ve} \) associated to \( v \) has total \( v \)-weight \( d_v \) (and depends on the choice of \( \alpha_{vw} \)).

Next, choose one admissible monomial \( M_{ve} \) for each edge at \( v \) and consider \( \delta_v - 2 \) equations associated to \( v \) by equating to 0 some \( \mathbb{C} \)-linear combinations of
these monomials:
\[
\sum_\mathcal{e} a_{\mathcal{e}} M_{\mathcal{e}} = 0, \quad i = 1, \ldots, \delta_v - 2.
\]
Repeating for all nodes, we get a total of \( n - 2 \) equations. If the coefficients \( a_{\mathcal{e}} \) of the equations are “sufficiently general,” we say that the resulting system of \( n - 2 \) equations is of splice type.

Sufficiently general simply means that for every \( v \), all maximal minors of the \( (\delta_v - 2) \times \delta_v \) matrix \((a_{\mathcal{e}})\) of coefficients should be non-singular. By applying row operations to such a matrix (taking linear combinations of the equations) one can always put the \( (\delta_v - 2) \times \delta_v \) coefficient matrix in the form
\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & a_1 & b_1 \\
0 & 1 & \ldots & 0 & a_2 & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & a_{\delta_v - 2} & b_{\delta_v - 2}
\end{pmatrix}
\]
so we will often assume we have done so. In this way, the defining equations are sums of three monomials. The “sufficiently general” condition is then \( a_i b_j - a_j b_i \neq 0 \) for all \( i \neq j \), and all \( a_i \) and \( b_i \) nonzero.

**Example 1.** Assume \( \Delta \) has one node, of valency \( n \). There is no semigroup condition. There is only one admissible monomial for each edge, namely \( z_d^j \), where \( d_j \) is the weight on the edge. Our equations are thus of Brieskorn-Pham type:
\[
\sum_{j=1}^n a_{ij} z_{d_j}^j = 0, \quad i = 0, \ldots, n - 2.
\]
The “sufficiently general” condition is then the well-known condition (due to H. Hamm [9]) for the system of \( n - 2 \) equations to have an isolated singularity. Thus, for a splice diagram with one node, “splice type” is equivalent to isolated Brieskorn-Pham complete intersection.

**Example 2.** For the \( \Delta \) of the example at the start of Section 1 we associate variables \( z_1, \ldots, z_4 \) to the leaves as follows:

\[
\Delta = \begin{array}{cccc}
z_1 & z_2 & z_3 & z_4 \\
z_2 & 3 & 7 & 11 \\
z_3 & 2 & 5 & 11 \\
z_4 & 2 & 11 & 7 \\
\end{array}
\]

The admissible monomials for the left node are \( z_1^2, z_3^1, \) and \( z_3 z_4 \). The admissible monomials for the right node are \( z_3^2, z_4^2 \), and \( z_1 z_2^2 \) or \( z_1^2 z_2 \) (since \( 11 = \alpha \cdot 3 + \beta \cdot 2 \) has solutions \((1, 4)\) and \((3, 1)\)). Thus the system of equations might be
\[
\begin{align*}
z_1^2 + z_2^3 + z_3 z_4 &= 0, \\
z_3^2 + z_4^2 + z_1 z_2^2 &= 0.
\end{align*}
\]
This system is always of “splice type” by our comments above.

The importance of splice type singularities is indicated by a result we prove elsewhere.

**Theorem 2.1 ([25]).** A system of equations of splice type defines an isolated complete intersection surface singularity whose link is the homology sphere \( \Sigma \) defined by the splice diagram \( \Delta \), and whose resolution graph is therefore the corresponding resolution diagram.
Moreover, the curve $z_i = 0$ cuts out in $\Sigma$ the knot corresponding to the $i$-th leaf of $\Delta$.

Equisingular deformations of systems of equations of splice type should come from adding terms of greater or equal weight with respect to the vertex weights to each equation. If only greater weight is allowed we speak of a higher weight deformation. See [25] for a fuller discussion.

One could in fact expand the definition of splice type singularities slightly, to include (for fixed $v$) suitable linear combinations of all possible admissible monomials associated to edges at $v$. But, up to higher weight deformations, this adds no generality. Also, if we change our choice of admissible monomials for the edges at each node, then we only change our splice type singularities up to higher weight deformation.

**Conjecture 2** (Splice Type Conjecture). Any Gorenstein surface singularity with integral homology sphere link is a higher weight deformation of a complete intersection of splice type.

Implicit in this conjecture is Conjecture 1 on the necessity of the semigroup condition.

A variant of the splice diagram yields a more familiar object. Let $\Delta$ be a splice diagram satisfying the semigroup conditions, and choose a distinguished leaf $w'$. Attach a variable $z_w$ to each leaf $w \neq w'$. Now, for each vertex $v$ of $\Delta$, form the same equations as before, except that one does not consider the edge in the direction of $w'$. (One is in general eliminating more monomials than simply setting $z_{w'} = 0$ in our previous splice diagram equations.) There is now one equation less than there are variables. Note that the edge-weights in the direction of $w'$ now play no role and can be discarded. We claim these equations generate a complete intersection curve, and this curve is the monomial curve associated to a semigroup $\Gamma'$. To describe this we first briefly recall some terminology about semigroups (see Section 6 for more details).

The semigroups arising in this paper are always numeric semigroups, that is subsemigroups $\Gamma$ of $\mathbb{N} = \mathbb{Z}_{\geq 0}$ for which $\mathbb{N} - \Gamma$ is finite. The conductor $c(\Gamma)$ is the smallest $c \geq 0$ so that $\gamma \geq c$ implies $\gamma \in \Gamma$. The semigroup ring $\mathbb{C}[t^\Gamma]$, or monomial curve associated to $\Gamma$, is the graded subalgebra of $\mathbb{C}[t]$ generated by $t^\gamma, \gamma \in \Gamma$. $\Gamma$ is called a complete intersection semigroup if $\mathbb{C}[t^\Gamma]$ is a graded complete intersection.

In our situation of a splice diagram $\Delta$ satisfying the semigroup conditions with distinguished leaf $w'$, the semigroup $\Gamma'$ is the semigroup generated by $\ell_{w'w}$ over all leaves $w \neq w'$. We will see in section 6 that $\Gamma'$ is a complete intersection semigroup and that the curve described above is isomorphic to the monomial curve $\mathbb{C}[t^{\Gamma'}]$.

**Example 3.** Consider the splice diagram at the beginning of Section 1, and let $w'$ be the lower left leaf. In the modified splice diagram, the weights 3 and 11 are removed. Denote the three leaves by $w_i, i = 1, 3, 4$, starting at the upper left and going counterclockwise; the corresponding variables by $z_i$; and the two nodes by $v$ and $v'$. Then the equations at $v$ resp. $v'$ could be $z_1^2 + az_3z_4 = 0$ and $z_3^5 + bz_4^2 = 0$. The semigroup $\Gamma'$ is $\Gamma' = \mathbb{N}(7, 4, 10)$ and, if we choose $a = b = -1$, the curve can be parameterized as $(z_1, z_3, z_4) = (t^7, t^4, t^{10})$.

In terms of Theorem 2.1, the significance of this curve is that if $w'$ is the $i$-th leaf of $\Delta$ then this curve, or an equisingular deformation of it, arises as the curve cut out by the hyperplane $z_i = 0$. 

A leaf $w'$ of a splice diagram $\Delta$ always represents a knot in the corresponding homology sphere, and this knot is a fibered knot (see §11 of [7]). If the homology sphere is given as a link of a splice type singularity as above, then this knot is the link of the curve cut out by a coordinate hyperplane $z_i = 0$ (and the fibration can be given by the usual Milnor fibration $z_i/|z_i|$). The first Betti number of its fiber is the Milnor number of the knot. We recall that even without the semigroup condition, we have:

**Theorem 2.2** ([7], §11). The Milnor number of the above knot is

$$1 + \sum_{v \neq w'} (\delta_v - 2)\ell_{v w'}.$$ 

If the link is given by splice type equations, then the theory of curve singularities implies that this number equals the conductor of the above semigroup $\Gamma'$, as can be confirmed by computation of the conductor (Theorem 6.1).

### 3. Milnor fibers

Suppose $\Sigma$ is the link of an isolated singularity at 0 of a complete intersection surface $X = f^{-1}(0)$, where $f$ is a map $f = (f_1, \ldots, f_{n-2}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-2}, 0)$. The Milnor fiber is the manifold $F := f^{-1}(\delta) \cap B(\epsilon)$ where $B(\epsilon)$ is a sufficiently small ball about 0 and $\delta$ is a general point of $\mathbb{C}^{n-2}$ very close to the origin. It is a smooth simply-connected piece of complex surface with boundary $\Sigma$; it has a symmetric intersection pairing on the second homology group, whose rank $b_2(F)$ is usually denoted by $\mu$. The Casson Invariant Conjecture says that when $\Sigma$ is a homology sphere, $\text{sign}(F) = 8\lambda(\Sigma)$, where $\lambda(\Sigma)$ is the Casson invariant.

The Casson invariant of $\Sigma$ is not hard to compute, and the hurdle in confirming this conjecture for any particular example is to understand $F$ well enough to compute $\text{sign}(F)$. This has been done for Brieskorn-Pham complete intersections. Thus, the conjecture could be verified in this case—a one-node splice diagram (see [22], which also proves a few other cases).

Now suppose the equations $f_i(z_1, \ldots, z_n) = 0$, $i = 1, \ldots, n-2$, are of splice type as above, corresponding to a splice diagram $\Delta$. Thus the curve $z_j = 0$ cuts out in $\Sigma$ the knot $K_j$ corresponding to the $j$-th leaf of $\Delta$. The link $(\Sigma, K_j)$ is a fibered link whose fiber $G_j$ can also be seen as the Milnor fiber of the singularity at 0 of the complete intersection curve $(f_1, \ldots, f_{n-2}, z_j)^{-1}(0)$. The topology of this fiber and its embedding in $\Sigma$ can be described by gluing together Milnor fibers of appropriate links in the splice components of $\Sigma$ (see [7]).

We shall describe a conjectural iterative description of $F$ in terms of the Milnor fibers of simpler complete intersection surface singularities and fibers $G_j$ as above lying in their boundaries.

Thus consider $\Sigma$ as the splice $\Sigma = \Sigma_1 K_1 K_2 \Sigma_2$ of two homology spheres determined by cutting $\Delta$ at an edge to form two diagrams. It is easy to see that these two diagrams $\Delta_1$ and $\Delta_2$ also satisfy the semigroup condition so $\Sigma_1$ and $\Sigma_2$ are both complete intersection singularity links given by equations of splice type. They thus have Milnor fibers, which we shall call $F_1$ and $F_2$, with $\partial F_1 = \Sigma_1$.

Let $G_1 \subset \Sigma_1$ be the fiber for the knot $(\Sigma_1, K_1)$. We may push the embedding $G_1 \to F_1$ inside $F_1$ by a normal vector-field to obtain a proper embedding $G_1 \to F_1$ (that is, an embedding with $\partial G_1 = G_1 \cap \partial F_1$, transverse intersection) and then
extend to an embedding $G_1 \times D^2 \to F_1$ of a tubular neighborhood of $G_1$. We similarly construct an embedding $D^2 \times G_2 \to F_2$.

Denote

$$F_1^u := F_1 - (G_1 \times \hat D^2), \quad F_2^o := F_2 - (\hat D^2 \times G_1),$$

so $\partial F_1^u$ is the union of $G_1 \times S^1$ and the exterior (complement of an open tubular neighborhood) of the knot $K_1 \subset \Sigma_1$, and similarly for $\partial F_2^o$.

**Conjecture 3** (Milnor Fiber Conjecture). $F$ is homeomorphic to the result $\mathcal{F}$ of pasting:

$$\mathcal{F} := F_1^u \cup_{G_1 \times S^1} (G_1 \times G_2) \cup S^1 \times G_2 \cup F_2^o,$$

where we identify $G_1 \times S^1$ with $G_1 \times \partial G_2$ and $S^1 \times G_2$ with $\partial G_1 \times G_2$.

By Milnor [17] and Hamm [9], $F$, $F_1$, $F_2$ are simply connected 4-manifolds which are homotopy equivalent to 2-complexes and thus have reduced homology only in dimension 2. We show that $\mathcal{F}$ has the nice properties we would like $F$ to have.

**Theorem 3.1.** $\partial \mathcal{F} = \Sigma$ and $\mathcal{F}$ is simply connected and homotopy equivalent to a 2-complex. Moreover,

$$H_2(\mathcal{F}) \cong H_2(G_1 \times G_2) \oplus H_2(F_1) \oplus H_2(F_2)$$

$$= (H_1(G_1) \otimes H_1(G_2)) \oplus H_2(F_1) \oplus H_2(F_2).$$

with maps induced by inclusions, so

$$\text{sign}(\mathcal{F}) = \text{sign}(F_1) + \text{sign}(F_2).$$

**Corollary 3.2.** The Milnor Fiber Conjecture (Conjecture 3) implies the Casson Invariant Conjecture for complete intersection singularities of splice type.

**Proof.** The theorem and Conjecture 3 imply that signature of Milnor fiber is additive under splicing. The Casson invariant is additive for splicing. The Casson Invariant Conjecture is known for Brieskorn-Pham complete intersections (the one-node case). \qed

**Proof of Theorem 3.1.** The fact that $\partial \mathcal{F} = \Sigma$ is immediate from the construction. For the rest of this proof it is convenient to have a different description of $\mathcal{F}$.

Consider $G_i$ embedded in $\Sigma_i$ and let $N_i \subset \Sigma_i = \partial F_i$ be a tubular neighborhood of $G_i$ in $\Sigma_i$, so $N_i \cong G_i \times I$. Note that $\partial(G_1 \times G_2) = (G_1 \times K_2) \cup (K_1 \times G_2)$, so we can also embed $N_i$ in $\partial(G_1 \times G_2)$ as $G_1 \times I \subset G_1 \times K_1$, and similarly for $N_2$. We claim:

$$\mathcal{F} \cong F_1 \cup_{N_1} (G_1 \times G_2) \cup_{N_2} F_2. \tag{3}$$

Indeed, to turn our previous description of $\mathcal{F}$ into this one, connect the proper embedding $G_i \subset F_i$ to the embedding $G_i \subset \partial F_i$ by a “strip” $G_i \times I$ and remove a tubular neighborhood of this strip from $F_i^u$ and glue it onto $F_1 \times F_2$ instead. The result of removing it from $F_i^u$ is something homeomorphic to $F_i$, while, when glued to $F_1 \times F_2$ it is just a collar on part of the boundary and does not change the homeomorphism type of $F_1 \times F_2$.

Consider, therefore, $\mathcal{F}$ as in equation (3). By shrinking slightly the regions $N_i$ along which the $F_i$ are glued to $G_1 \times G_2$ we can make them disjoint in $\partial(G_1 \times G_2)$ without changing the homotopy type (or even homeomorphism type) of $F_1 \cup_{N_1} (G_1 \times G_2) \cup_{N_2} F_2$. Then $(G_1 \times G_2) \cap (F_1 \cup F_2)$ consists of the disjoint union of $N_1$
and \( N_2 \). The Meyer-Vietoris sequence for the decomposition \((G_1 \times G_2) \cup (F_1 \cup F_2)\) then easily yields that the inclusions induce an isomorphism
\[
H_2(\overline{F}) \cong H_2(G_1 \times G_2) \oplus H_2(F_1) \oplus H_2(F_2)
\]
as desired.

The fact that \( \overline{F} \) is simply connected is an easy application of the Van Kampen theorem. The fact that \( \overline{F} \) is homotopy equivalent to a 2-complex can be seen by replacing \( G_1 \) and \( G_2 \) by one-dimensional spines \( S_1 \) and \( S_2 \), say, replacing \( F_1 \) and \( F_2 \) by 2-dimensional spines \( T_1 \) and \( T_2 \), and then gluing \( S_1 \times S_2 \) to \( T_1 \) and \( T_2 \) by means of mapping cylinders of appropriate maps \( S_i \to T_i \). □

Recall that the geometric genus \( p_g(X,o) \) of a singularity is \( \dim H^1(Y,\mathcal{O}) \), where \( Y \to X \) denotes a resolution of the singularity. In general, it is not topologically determined by the link of \( X \), but the Casson Invariant Conjecture (second version) says that it should be for Gorenstein singularities with homology sphere links. The following theorem says what the Casson Invariant Conjecture implies about the behavior of various invariants under splicing. Item (3) of this theorem provided part of the motivation for the above construction of \( \overline{F} \) for the Milnor Fiber Conjecture.

**Theorem 3.3.** Let \( X \) be a complete intersection with homology sphere link, with Milnor fiber \( F \); and suppose its link is spliced from links of two singularities \( X_1, X_2 \), with Milnor fibers \( F_1, F_2 \). Assume the Casson Invariant Conjecture for \( X_1 \) and \( X_2 \). Then the following statements are equivalent:

1. The Casson Invariant Conjecture holds for \( X \).
2. We have \( \text{sign}(F) = \text{sign}(F_1) + \text{sign}(F_2) \).
3. With \( G_1, G_2 \) as above, we have \( b_2(F) = b_2(F_1) + b_2(F_2) + b_1(G_1)b_1(G_2) \), where \( b_1 \) is Betti number.
4. The geometric genus satisfies \( p_g(X) = p_g(X_1) + p_g(X_2) + \frac{1}{2}b_1(G_1)b_1(G_2) \).

Moreover, these invariants of \( X \) are then topologically determined by the link.

**Proof.** The equivalence of (1) and (2) has already been discussed, so we prove the equivalence of (2), (3), and (4).

Formulas of H. Laufer and A. Durfee imply that the geometric genus of \( X \), and the signature and second Betti number \( \mu \) of the Milnor fiber, are explicitly related by topological invariants of the link (see, e.g., [28].) Let \( Y \to X \) denote a good resolution, and \( c_1^2 \) and \( c_2 \) the characteristic Chern numbers of \( Y \) (also known as \( K \cdot K \) and \( \chi(Y) \), where \( \chi \) is topological Euler characteristic). Then these Chern numbers are determined by the resolution dual graph, and their sum \( c_1^2 + c_2 \) is independent of the resolution, hence depends only on the link. We define
\[
C(\Delta) = c_1^2 + c_2 - 1,
\]
the notation indicating that this number depends only on the splice diagram \( \Delta \). Then the aforementioned formulas may be written
\[
\mu = 12p_g + C(\Delta) \quad \text{(Laufer)}
\]
\[
3\text{sign}(F) = -2\mu - C(\Delta) \quad \text{(Durfee)}
\]
(In general Durfee’s formula has an extra \( 3b_1(Y) \) on the right, which vanishes in our case.) Eliminating \( \mu \), these formulas imply
\[
\text{sign}(F) = -8p_g - C(\Delta),
\]
proving the equivalence of the two formulations of the Casson Invariant Conjecture in the Introduction (for complete intersections). Moreover, it follows that the equivalence of (2), (3), and (4) of Theorem 3.3 reduce to the formula of the following theorem, which will therefore complete the proof. □

**Theorem 3.4.** In the above notation, even if $\Delta$ does not satisfy the semigroup condition we have

$$C(\Delta) - C(\Delta_1) - C(\Delta_2) = -2b_1(G_1)b_1(G_2).$$

This theorem involves computing $c_1^2$ and $c_2$ of the resolution in terms of the splice diagram, which is of interest in its own right, so we devote the next section (Section 4) to its proof.

If any one of the analytic invariants sign$(F)$, $\mu$, and $p_g(X)$ is a topological invariant, then they all are, by the above formulas. The Casson Invariant Conjecture gives a topological description of $p_g$ and sign$(F)$.

Suppose $(X, o)$ is a Gorenstein surface singularity whose homology sphere link has one node; thus, its link is $\Sigma(p_1, \ldots, p_n)$. The Casson Invariant Conjecture for $X$ is equivalent to the assertion that $p_g(X) = p_g(V(p_1, \ldots, p_n))$. But this latter condition is well-known to be equivalent to the statement that $X$ admits an equisingular, simultaneous resolution degeneration to $V$ (see, e.g., [27] (6.3) for a convenient proof). In other words, we could conclude that $X$ is an equisingular deformation of a splice type singularity, as was mentioned in the Introduction. We suspect a similar result is true in the general case. But, even in case $X$ is a hypersurface singularity with link $\Sigma(p, q, r)$, we do not know a proof. As we mentioned in the Introduction, there are a few very non-trivial cases worked out by A. Némethi ([18]).

4. **Canonical divisor of a resolution**

This section is devoted to proving Theorem 3.4. We start by computing the rational canonical divisor for an arbitrary resolution of an isolated surface singularity.

Suppose we have a good resolution of an isolated surface singularity. Denote the exceptional curves by $E_i$, $i = 1, \ldots, n$. For each $i$ let $\delta_i$ be the number of intersection points of $E_i$ with other $E_j$’s and $E_i^0$ be $E_i$ with these intersection points removed. Denote $\chi_i = \chi(E_i^0) = \chi(E_i) - \delta_i$ ($\chi$ is Euler characteristic).

Let $K$ be the (rational) canonical divisor, defined by the adjunction formula

$$K \cdot E_i = -\chi(E_i) - E_i \cdot E_i.$$  

Let

$$D := -K - E, \quad \text{where} \quad E = \sum_{i=1}^{n} E_i$$

and suppose

$$D = \sum_{i=1}^{n} k_i E_i.$$  

Then the adjunction formula becomes $D \cdot E_j = \chi(E_j) - \delta_j = \chi_j$, so

$$k_i = -\sum \ell_{ij} \chi_j, \quad \text{where} \quad (\ell_{ij}) = (-E_i \cdot E_j)^{-1} \quad (\text{matrix inverse}).$$
\[ K = -D - E = \sum_i (-k_i - 1)E_i = \sum_i \left( \sum_j \ell_{ij} \chi_j - 1 \right)E_i \]

so
\[
K \cdot K = \left( \sum_i \left( \sum_j \ell_{ij} \chi_j - 1 \right)E_i \right) \cdot \left( \sum_k \left( \sum_l \ell_{kl} \chi_l - 1 \right)E_k \right)
\]
\[
= \sum_{i,j,k,l} \ell_{ij} \ell_{kl} (E_i \cdot E_k) \chi_j \chi_l - \sum_{i,k,l} \ell_{kl} (E_i \cdot E_k) \chi_l - \sum_{i,j,k} \ell_{ij} (E_i \cdot E_k) \chi_j
\]
\[
+ \sum_{i,k} E_i \cdot E_k
\]
\[
= - \sum_{i,j} \ell_{ij} \chi_i \chi_j + \sum_i \chi_i + \sum_i \chi_i + \left( \sum_i (E_i \cdot E_i + 2 \sum_{i<j} E_i \cdot E_j) \right)
\]
\[
= - \sum_{i,j} \ell_{ij} \chi_i \chi_j + 2 \chi (\bigcup_i E_i) + \sum_i E_i \cdot E_i .
\]

We note that our notation \( k_i \) and \( \ell_{ij} \) is consistent with the notation in the Appendix (Section 9). Summarizing:

**Proposition 4.1.** For any good resolution of an isolated surface singularity the divisor \( D = -K - E \) is given by
\[
D = \sum_i k_i E_i \quad \text{with} \quad k_i = - \sum_j \ell_{ij} \chi_j
\]

where \( (\ell_{ij}) = (-E_i \cdot E_j)^{-1} \) (matrix inverse). Also,
\[
c_1^2 + c_2 = - \sum_{ij} \ell_{ij} \chi_i \chi_j + 3c_2 + \sum E_i \cdot E_i .
\]

To apply this to prove Theorem 3.4 we now restrict to the case of a singularity with homology sphere link given by a splice diagram \( \Delta \). Then \( \chi_i = 2 - \delta_i \) which vanishes except at nodes and ends of the plumbing graph, so we only need to know \( \ell_{ij} \) when \( i \) and \( j \) index nodes or ends. By Theorem 9.1, this is as follows. If \( i \neq j \) then \( \ell_{ij} \) is the product of the splice diagram weights adjacent but not on the path from \( i \) to \( j \) in \( \Delta \). If \( i = j \) then

- If \( i \) is a node then \( \ell_{ii} \) is the product of weights at that node.
- If \( i \) is a leaf adjacent to a node with weights \( p_0, \ldots, p_n \) with \( p_0 \) on the edge to \( i \) then \( \ell_{ii} = [p_1 \ldots p_n/p_0] \).

A simple matrix calculation shows that \( -\ell_{ii} \) is the weight one would have to put on a new vertex attached to vertex \( i \) by a new edge, to get an extended plumbing diagram of determinant 0 (this is computed in [7] and gives an alternative proof of the description of \( \ell_{ii} \)).

Now let \( C(T) \) denote \( c_1^2 + c_2 - 1 \) computed for a plumbing graph \( T \). We want to compute the effect of splicing on \( C \). So suppose that \( T \) is the result of splicing diagrams \( T_1 \) and \( T_2 \). Let \( I_1 \) and \( I_2 \) be index sets for the nodes and leaves of \( T_1 \) and \( T_2 \) with \( 0 \in I_1 \) and \( 1 \in I_2 \) representing the leaves at which we splice. Then \( \hat{I} = I_1 \cup I_2 - \{0,1\} \) is the index set for nodes and leaves of \( T \).

Let \( \hat{T}_1 \) be the result of extending \( T_1 \) at vertex 0 by a vertex with weight \(-\ell_{00}(T_1)\) and similarly for \( \hat{T}_2 \). Let \( \hat{T} \) be the result of attaching \( \hat{T}_1 \) to \( \hat{T}_2 \) by an edge joining the new vertices. Then in [7] it is shown that \( T \) results from \( \hat{T} \) by a sequence of

\[
K = -D - E = \sum_i (-k_i - 1)E_i = \sum_i \left( \sum_j \ell_{ij} \chi_j - 1 \right)E_i
\]
Thus
\[ \sum \] Moreover each blow-down reduces \( \sum E_i \cdot E_i \) by 3 and the 0-absorption does not change it, so
\[ \sum E_i \cdot E_i = \sum_{T_1} E_i \cdot E_i - \ell_{00} + \sum_{T_2} E_i \cdot E_i - \ell_{11} - 3r . \]
Thus
\[ C(T) = - \sum_{i,j \in I} \ell_{ij} \chi_i \chi_j + 3c_2(T) + \sum_T E_i \cdot E_i - 1 \]
\[ = - \left( \sum_{i,j \in I_1} \ell_{ij} \chi_i \chi_j - \ell_{00} - 2 \sum_{i \in I_1 \cap \{0\}} \ell_{0i} \chi_i + \sum_{i,j \in I_2} \ell_{ij} \chi_i \chi_j - \ell_{11} - 2 \sum_{j \in I_2 \cap \{1\}} \ell_{1j} \chi_j + 2 \sum_{j \in I_2 \cap \{0\}} \ell_{ij} \chi_i \chi_j \right) + 3(c_2(T_1) + c_2(T_2) - r - 1)
\[ + \sum_{T_1} E_i \cdot E_i + \sum_{T_2} E_i \cdot E_i - \ell_{00} - \ell_{11} - 3r - 1 \]
\[ = - \sum_{i,j \in I_1} \ell_{ij} \chi_i \chi_j + 2 \sum_{i \in I_1 \cap \{0\}} \ell_{0i} \chi_i - \sum_{i,j \in I_2} \ell_{ij} \chi_i \chi_j + 2 \sum_{j \in I_2 \cap \{1\}} \ell_{1j} \chi_j + 2 \sum_{j \in I_2 \cap \{0\}} \ell_{ij} \chi_i \chi_j 
\[ + 3c_2(T_1) + 3c_2(T_2)
\[ + \sum_{T_1} E_i \cdot E_i + \sum_{T_2} E_i \cdot E_i - 4 , \]
where the last equality uses the fact that for \( i \in I_1 \setminus \{0\} \) and \( j \in I_2 \setminus \{1\} \) one has \( \ell_{ij} = \ell_{0} \ell_{1j} \).

The above simplifies to
\[ C(T) = C(T_1) + C(T_2) - 2(\sum_{i \in I_1 \setminus \{0\}} \ell_{0i} \chi_i + 1) \left( - \sum_{j \in I_2 \setminus \{1\}} \ell_{1j} \chi_j + 1 \right) \]
\[ = C(T_1) + C(T_2) - 2\mu(T_1,0)\mu(T_2,1), \]
where \( \mu(T_i,i) \) is the Milnor number for the knot represented by vertex \( i \) in the homology sphere represented by \( T_i \), that is, the first Betti number of its fiber (it is a basic result of [7] that \( \sum_{i \in I_1 \setminus \{0\}} (\ell_{0i} \chi_i) \) is the Euler characteristic of the fiber in question). This completes the proof of Theorem 3.4. \( \square \)

5. Plane curves and their cyclic covers

Let \( (X, o) \) be a hypersurface singularity at the origin given by an equation in the form \( z^n + g(x, y) = 0 \) and suppose that its link is a homology sphere. The Casson Invariant Conjecture was proved in this case in [22] by a somewhat subtle calculation. In this section we will show that the Milnor Fiber Conjecture (Conjecture 3)
holds for these singularities, giving a more conceptual proof of the Casson Invariant Conjecture in this case. We must first explain how these hypersurface singularities fit the format of equations of splice type. In [22] we point out that if the link of $z^n + g(x, y) = 0$ is a homology sphere, then $g(x, y) = 0$ defines an irreducible plane curve singularity at the origin $o \in \mathbb{C}^2$. We therefore need to start by discussing how plane curve singularities in general, and irreducible plane curve singularities in particular, fit into the framework of our conjectures.

5.1. Non-minimal splice diagrams and plane curve singularities. Theorem 2.1 gives a general sufficient condition for a knot in a homology sphere to be realizable as the link of a germ $(Y, o) \subset (X, o)$ of a curve cut out by a single equation in a compete intersection surface. This has content also for non-minimal splice diagrams. For example, the splice diagram

$$
\Delta = \begin{array}{c}
\circ \\
2
\end{array} \begin{array}{c}
\circ \\
3
\end{array} \begin{array}{c}
\circ \\
1
\end{array} \begin{array}{c}
\circ \\
5
\end{array}
$$

is a non-minimal version of

so it represents the Seifert fibered homology sphere $\Sigma(2, 3, 5)$ (Poincaré’s dodecahedral space). The upper right vertex of $\Delta$ represents a particular knot in this homology sphere (a $(3, 2)$-cable on the degree 5 fiber of $\Sigma(2, 3, 5)$). Since $\Delta$ satisfies the semigroup condition, Theorem 2.1 tells us that this knot in $\Sigma(2, 3, 5)$ is the link of a complex curve singularity $(Y, o)$ cut out by a single equation in $(V(2, 3, 5), o)$. In fact, the splice type equations for $\Delta$ can be chosen as $z_1^2 + z_2^3 + z_5^3 = 0, z_1 + z_3^2 + z_4 = 0$, and the curve is then cut out by $z_4 = 0$. Eliminating $z_4$, the curve is cut out by the equation $z_1 + z_3^2 = 0$ in $V(2, 3, 5) = \{ (z_1, z_2, z_3) : z_1^2 + z_2^3 + z_3^3 = 0 \}$.

When $X$ is non-singular, that is, for a link of a plane curve singularity, the next proposition implies that we can always do the analogous thing. That is, for any irreducible plane curve singularity we will find splice type equations for $X (= \mathbb{C}^2)$ so that the curve $Y$ cut out by a coordinate function has the topology of the given plane curve. Corollary 8.2 below then says that the original plane curve singularity is an equisingular deformation of the one given by splice type equations.

**Proposition 5.1.** The splice diagram of any plane curve singularity satisfies the semigroup condition.

*Proof.* It is easy to see that the semigroup condition for the splice diagram of a reducible plane curve singularity follows from the semigroup condition for each of the subdiagrams for the irreducible branches of the plane curve. Thus we may assume that the germ $(\mathbb{C}^2, Y, o)$ is an irreducible germ. In this case the result is well known (see, e.g., Teissier’s appendix to [30]) but we give a proof in our language for completeness. By [7] the singularity is given by a splice diagram of the form:

$$
\begin{array}{c}
P_1 \\
q_1
\end{array} \begin{array}{c}
P_2 \\
q_2
\end{array} \ldots \begin{array}{c}
P_k \\
q_k
\end{array},
$$

where
where $\gcd(p_i, q_i) = 1$ for each $i$ and the positive edge determinant condition holds $(p_i > q_iq_{i-1}p_{i-1}$ for each $i > 1)$. Since this diagram may have arisen as a subdiagram of a diagram for a plane curve with several branches, we cannot assume that it is a reduced diagram, so some of the $q_j$ may equal 1.

The only non-trivial cases of the semigroup condition for this diagram are:

$$p_{j+1} \in S_j := \mathbb{N}(q_1q_2 \cdots q_j, p_1q_2 \cdots q_j, \ldots, p_{j-1}q_j, p_j)$$

for each $j = 1, \ldots, k - 1$. Since $p_{j+1} > p_jq_jq_{j+1} \geq p_jq_j$ it suffices to show that the conductor $\mu_j$ of this semigroup satisfies $\mu_j \leq p_jq_j$. Proposition 6.3 of section 6 implies $\mu_j = q_j(\mu_{j-1} - 1) - p_j + p_jq_j + 1$ (or use Theorem 2.2 and its following paragraph). The desired inequality is now a trivial induction. □

This gives a new way to find an equation for a plane curve singularity of given topology: start with the equations of splice type and then eliminate variables to obtain an equation in $\mathbb{C}^2$. To describe this in detail, let us assign variables to the leaves of our splice diagram as follows:

$$\begin{array}{c}
z_0 \stackrel{p_1}{\rightarrow} z_1 \quad \stackrel{1}{\rightarrow} \quad z_2 \quad \stackrel{1}{\rightarrow} \quad \cdots \quad \stackrel{1}{\rightarrow} \quad z_k \quad \stackrel{1}{\rightarrow} \quad z_{k+1}.
\end{array}$$

The only admissible monomial for the outgoing edge to the right at the $j$-th node is $z_{j+1}$. Thus the general system of equations of splice type can be written

$$z_2 = a_1z_1^{q_1} + a_0z_0^{p_1},$$
$$z_3 = a_2z_2^{q_2} + g_2(z_0, z_1),$$
$$\ldots \ldots \ldots$$
$$z_k = a_{k-1}z_{k-1}^{q_{k-1}} + g_{k-1}(z_0, \ldots, z_{k-2}),$$
$$z_{k+1} = a_kz_k^{q_k} + g_k(z_0, \ldots, z_{k-1}),$$

where $g_j(z_0, \ldots, z_{j-1})$ is a multiple of an admissible monomial for the left edge at the $j$-th node, that is, a monomial of the form $z_0^{a_0} \cdots z_{j-1}^{a_{j-1}}$ with

$$a_0q_1 + q_1z_{j-1} + a_1p_1q_2 \cdots q_{j-1} + \cdots + a_{j-2}p_{j-2}q_j + a_{j-1}p_j - 1 = p_j.$$  

We now successively substitute each of the above equations into the next to put them in the form:

$$z_2 = a_1z_1^{q_1} + a_0z_0^{p_1},$$
$$z_3 = a_2(a_1z_1^{q_1} + a_0z_0^{p_1})^{q_2} + g_2(z_0, z_1) =: f_2(z_0, z_1),$$
$$\ldots \ldots \ldots$$
$$z_{k+1} = a_kf_k(z_0, z_1)g_k(z_0, z_1, \ldots, f_{k-2}(z_0, z_1)) =: f_k(z_0, z_1).$$

In terms of new coordinates, $x := z_0$, $y := z_1$, $Z_2 := z_2 - a_1z_1^{q_1} + a_0z_0^{p_1}$, \ldots, $Z_k := z_k - f_{k-1}(z_0, z_1)$, $Z_{k+1} := z_{k+1} - f_k(z_0, z_1)$ these equations become

$$Z_2 = Z_3 = \cdots = Z_k = Z_{k+1} = 0,$$

so our surface is the $(x, y)$-plane. Our plane curve is the curve cut out by the coordinate equation $z_{k+1} = 0$ which is $f_k(x, y) = 0$ in our new coordinates. Thus, if we write $f = f_k$, the equation of the plane curve is $f(x, y) = 0$.  

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We now address what the Milnor Fiber Conjecture says for this type of example. Our surface germ is a nonsingular point, and the Milnor fiber for a non-singular point is a disk, so the conjecture postulates a particular decomposition of $D^4$. Although it is rather trivial, it will be useful in the discussion of hypersurfaces of the form $z^n = g(x, y)$. We will therefore reserve the notations $G_1$ etc. of Conjecture 3 for that case and use primes (as in $G'_1$ etc) to distinguish the ingredients involved in the present discussion.

Suppose therefore that we have decomposed our splice diagram as the splice of two diagrams:

$$
\begin{align*}
\text{left diagram} & : p_1 \quad p_2 \quad p_r \quad p_{r+1} \quad p_k \quad \ldots \ldots \\
\text{right diagram} & : q_1 \quad q_2 \quad q_r \quad q_{r+1} \quad q_k \quad \ldots \ldots
\end{align*}
$$

The left diagram represents a plane curve whose Milnor fiber we will denote by $G'_1 \subset S^3 = \partial D^4$. The right diagram is a non-reduced diagram for the trivial knot in $S^3$ so its Milnor fiber is $G'_2 = D^2 \subset S^3 = \partial D^4$.

Let $(F'_1)\circ$ be the result of removing from $D^4$ a tubular neighborhood of $G'_1$ pushed inside to a proper embedding $G'_1 \subset D^4$. Let $(F'_2)\circ$ be the result of removing a tubular neighborhood of a proper embedding $D^2 \subset D^4$. Note that $(F'_2)\circ \cong S^1 \times D^3$.

The Milnor Fiber Conjecture says that the result of the pasting:

$$(4) \quad (F'_1)\circ \cup (G'_1 \times D^2) \cup (F'_2)\circ$$

should be $D^4$. This is indeed clear, since, starting with $(F'_1)\circ$, the first pasting clearly gives $D^4$ back, while the second just pastes a collar onto a portion of the boundary of this $D^4$.

5.2. The hypersurface $z^n + g(x, y) = 0$. As already mentioned, if the link of $z^n + g(x, y) = 0$ is a homology sphere, then $g(x, y) = 0$ defines a plane curve singularity at $(0, 0) \in \mathbb{C}^2$ which is irreducible. Its splice diagram therefore has the form

$$
\begin{align*}
\text{splice diagram} & : p_1 \quad p_2 \quad p_r \quad p_k \quad \ldots \ldots \\
\end{align*}
$$

where $\gcd(p_i, q_i) = 1$ for each $i$ and the positive edge determinant condition holds ($p_i > q_i q_{i-1} p_{i-1}$ for each $i > 1$). Moreover, given an irreducible plane curve singularity as above, we showed in [22] that the hypersurface singularity defined by $z^n + g(x, y) = 0$ has homology sphere link if and only if $n$ is relatively prime to all the $p_i$ and $q_i$, and the splice diagram for the link of this singularity is then

$$(5) \quad \begin{align*}
\text{splice diagram} & : p_1^n \quad p_2^n \quad p_r^n \quad p_k^n \quad \ldots \ldots \\
\end{align*}
$$

We now show that the splice diagram equations for this splice diagram reduce to the equation $z^n = f(x, y)$, with $f$ as in the previous subsection (Corollary 8.2 below shows that the original $z^n = g(x, y)$ is an equisingular deformation of this).
We assign variables to the leaves of the splice diagram (5) as follows:

\[
\begin{array}{cccc}
Z & P_1 & n & P_2 & n & \cdots & P_k & n & z \\
\l & q_1 & \r & q_2 & \r & \cdots & q_k & \r & \\
\l & y & z_1 & z_2 & \cdots & z_k & \r \\
\end{array}
\]

The only admissible monomial for the outgoing edge to the right at the \(j\)-th node is \(z_{j+1}\) if \(j < k\) and \(z^n\) if \(j = k\). Thus the general system of equations of splice type can be written

\[
\begin{align*}
z_2 &= a_1z_1^{q_1} + a_0z_0^{p_1} \\
z_3 &= a_2z_2^{q_2} + g_2(z_0, z_1) \\
\vdots & \quad \vdots \quad \vdots \\
z_k &= a_kz_k^{q_k} + g_k(z_0, \ldots, z_{k-1}) \\
z^n &= a_kz_k^{q_k} + g_k(z_0, \ldots, z_{k-1}),
\end{align*}
\]

where the \(g_j(z_0, \ldots, z_k)\) are as before.

We again successively substitute each of these equations into the next to eliminate the variables \(z_2, z_3, \ldots, z_k\). To be precise, we first make these substitutions to put the equations in the form:

\[
\begin{align*}
z_2 &= a_1z_1^{q_1} + a_0z_0^{p_1} \\
z_3 &= f_2(z_0, z_1) \\
\vdots & \quad \vdots \quad \vdots \\
z_k &= f_k(z_0, z_1) \\
z^n &= f_k(z_0, z_1).
\end{align*}
\]

Recall our notation \(f = f_k\). In terms of new coordinates, \(x = z_0, y = z_1, z, Z_2 := z_2 - a_1z_1^{q_1} + a_0z_0^{p_1}, \ldots, Z_k := z_k - f_k(z_0, z_1)\), these equations become

\[
Z_2 = Z_3 = \cdots = Z_k = 0; \quad z^n = f(x, y).
\]

We are now ready to prove the main result of this section.

**Theorem 5.2.** Let \((X, o)\) be a hypersurface singularity at the origin given by an equation in the form \(z^n + g(x, y) = 0\) with homology sphere link. Then the Milnor Fiber Conjecture is true for \((X, o)\).

**Proof.** Suppose that we have a splice decomposition corresponding to the following decomposition of our splice diagram as the splice of two diagrams:

\[
\begin{array}{cccc}
Z & P_1 & n & P_2 & n & \cdots & P_r & n & P_{r+1} & n & \cdots & P_k & n & z \\
\l & q_1 & \r & q_2 & \r & \cdots & q_r & \r & q_{r+1} & \r & \cdots & q_k & \r & \\
\l & y & z_1 & z_2 & \cdots & z_r & \cdots & z_{k-1} & \r \\
\end{array}
\]

We wish to show that the Milnor fiber \(F\) for \(z^n = g(x, y)\) is obtained by the construction \(F_1 \cup_{N_i} (G_1 \times G_2) \cup_{N_i} F_2^o\) of Conjecture 3, where \(F_1\) and \(F_2\) are Milnor fibers for the two splice components, \(G_1\) and \(G_2\) are fibers in the links of the two splice components for the knots along which we splice, and \(F_i^o\) is the result of removing a tubular neighborhood of a properly embedded \(G_i\) in \(F_i\).

In [21] (see also [12]) it is shown that the Milnor fiber \(F\) is obtained by pushing a Milnor fiber \(G \subset S^3 = \partial D^4\) inside \(D^4\) so that it is properly embedded (that is,
∂G = G ∩ ∂D^4), and then taking the n-fold branched cyclic cover of D^4, branched along this embedding of G.

We need to understand the placement of G with respect to the decomposition of D^4 of equation (4). On taking the n-fold branched cover we will see that we get the desired decomposition of F.

According to [7] the fiber G decomposes according to the splice diagram into q_{r+1} \ldots q_k parallel copies of the Milnor fiber G'_1 of the plane curve given by

\begin{align*}
& P_1 \quad 1 \quad P_2 \quad 1 \quad \ldots \ldots \quad P_r \quad 1 \\
& q_1 \quad q_2 \quad \ldots \ldots \quad q_r
\end{align*}

and one copy of the Milnor fiber of the plane curve corresponding to

\begin{align*}
& P_{r+1} \quad 1 \quad P_{r+2} \quad 1 \quad \ldots \ldots \quad P_k \quad 1 \\
& q_{r+1} \quad q_{r+2} \quad \ldots \ldots \quad q_k
\end{align*}

punctured q_{r+1} \ldots q_k times.

We can position G with respect to the decomposition of equation (4) so that it lies completely in (G'_1 × D^2) ∪ (F'_2)^o. It then intersects (G'_1 × D^2) in q_{r+1} \ldots q_k parallel copies of G'_1. Its intersection with (F'_2)^o is obtained as follows. First make the fiber G' of the knot represented by the right arrowhead of the splice diagram

\begin{align*}
& P_{r+1} \quad 1 \quad P_{r+2} \quad 1 \quad \ldots \ldots \quad P_k \quad 1 \\
& q_{r+1} \quad q_{r+2} \quad \ldots \ldots \quad q_k
\end{align*}

properly embedded in D^4 and transverse to the properly embedded version of the fiber D^2 of the unknot represented by the left arrowhead. Then remove the tubular neighborhood of the latter. Using [21], the n-fold cyclic cover of D^4 along G' is the Milnor fiber for the surface singularity with diagram:

\begin{align*}
& P_{r+1} \quad n \quad P_{r+2} \quad n \quad \ldots \ldots \quad P_k \quad n \\
& q_{r+1} \quad q_{r+2} \quad \ldots \ldots \quad q_k
\end{align*}

Moreover, the embedded D^2 ⊂ D^4 lifts in this cover to copy of the fiber for the knot represented by the left-most vertex.

It follows that the decomposition of equation (4) lifts to give the desired decomposition of F, as desired.

\[ \square \]

6. Numerical semigroups and monomial curves

In this section we develop some results about semigroups and their associated curves that are needed in the proofs of Theorems 2 and 3 of the Introduction.

As mentioned in Section 2, the semigroups we consider are always numeric semigroups, that is, subsemigroups Γ of N = Z_≥0 for which N − Γ is finite. The semigroup ring \( \mathbb{C}[t^\Gamma] \), or monomial curve associated to Γ, is the graded subalgebra of \( \mathbb{C}[t] \) generated by \( t^\gamma, \gamma \in \Gamma \). We briefly collect some known facts and terminology (e.g., [6, 10, 11, 29]).

The conductor \( c(\Gamma) \) is the smallest \( c \geq 0 \) so that \( \gamma \geq c \) implies \( \gamma \in \Gamma \). Γ is symmetric when \( \gamma \in \Gamma \) if and only if \( c(\Gamma) − 1 − \gamma \notin \Gamma \); equivalently, \( \mathbb{C}[t^\Gamma] \) is Gorenstein (see [11] Prop. 2.21). Since \( \gamma \) and \( c(\Gamma) − 1 − \gamma \) cannot both be in Γ, a
symmetric semigroup is maximal with given conductor. Classically an element of \( \mathbb{N} \) that is not in \( \Gamma \) is called a gap. The number of gaps is denoted \( \delta(\Gamma) \); clearly
\[
\delta(\Gamma) \geq c(\Gamma)/2, \\
\text{with equality if and only if } \Gamma \text{ is symmetric.}
\]

\( \Gamma \) is called a complete intersection semigroup if \( \mathbb{C}[t^\Gamma] \) is a graded complete intersection. A complete intersection semigroup is symmetric. \( \Gamma \) is a complete intersection semigroup if and only if it has a semigroup presentation of deficiency one (i.e., with one fewer relations than generators; see [10]). If \( \Gamma \) (complete intersection or not) has a semigroup presentation
\[
\Gamma = \langle x_1, \ldots, x_n : \sum_j a_{ij} x_j = \sum_j b_{ij} x_j, i = 1, \ldots, r \rangle
\]
with \( a_{ij}, b_{ij} \in \mathbb{N} \), then the monomial curve is presented as
\[
\mathbb{C}[z_1, \ldots, z_n]/\left( \prod_j z_j^{a_{ij}} - \prod_j z_j^{b_{ij}}, i = 1, \ldots, r \right).
\]

**Example.** Relatively prime \( p \) and \( q \) generate a complete intersection semigroup with conductor \((p-1)(q-1)\). This semigroup has semigroup presentation \( \langle x_1, x_2 : qx_1 = px_2 \rangle \). Its monomial curve \( \mathbb{C}[t^p, t^q] \) is presented as \( \mathbb{C}[z_1, z_2]/(z_1^q - z_2^p) \), with the isomorphism given by \( z_1 \mapsto t^p, z_2 \mapsto t^q \).

Let \((\Delta, w')\) be a finite rooted tree (tree with one vertex singled out as “root”), whose root vertex \( w' \) is of valency 1. We visualize it with the root vertex at the top, so “downward” means in the direction away from the root. We assume also that \( \Delta \) has positive integer weights on all edges other than the root edge and that the weights on the downward edges at each non-root vertex are pairwise coprime. For example, one obtains such a tree if one picks some leaf \( w' \) of a splice diagram as root, and then forgets all “far weights” of the splice diagram (splice diagram weights on the far end of edges from the point of view of \( w' \)).

In such a tree, the numbers \( \ell_{v,w} \) for \( v \neq w' \) are still defined (product of weights on edges directly adjacent to the shortest path from \( w' \) to \( v \)). We define the semigroup of \((\Delta, w')\) to be the semigroup
\[
\text{sg}(\Delta) = \text{sg}(\Delta, w') := \mathbb{N}\langle \ell_{w,w'} : w \text{ is a leaf of } \Delta \rangle
\]
(we use the shorter \( \text{sg}(\Delta) \) if the root vertex is clear). Each non-root vertex of \( \Delta \) cuts off a collection of subtrees below it. We say that \((\Delta, w')\) satisfies the semigroup condition if the weight on the root edge of every such subtree is in the semigroup of the subtree.

Define an invariant \( \mu(\Delta, w') \) by
\[
\mu(\Delta) = \mu(\Delta, w') := 1 + \sum_{v \neq w'} (\delta_v - 2) \ell_{w,v}.
\]

**Theorem 6.1.** Let \((\Delta, w')\) be a weighted rooted tree as above and \( \Gamma = \text{sg}(\Delta) \). Then
\[
2\delta(\Gamma) \leq \mu(\Delta),
\]
with equality if and only if \((\Delta, w')\) satisfies the semigroup condition, in which case \( \Gamma = \text{sg}(\Delta) \) is a complete intersection semigroup. (It follows that the same result holds with \( 2\delta(\Gamma) \) replaced by \( c(\Gamma) \).)
If \((\Delta, w')\) satisfies the semigroup condition we will describe the complete intersection equations; these equations will be associated to the nodes of \(\Delta\). We assign a variable \(z_j\) to each leaf \(w_j\) of \(\Delta\). The equations will generate the kernel of the map \(\mathbb{C}[z_1, \ldots, z_m] \to \mathbb{C}[t^\Gamma]\) given by \(z_j \mapsto t^{l_{w_j}'}\).

For a node \(v\) of the tree and a leaf \(w_j\) below it let \(l_{vw_j}'\) be the product of weights adjacent to the path from \(v\) to \(w_j\), excluding weights adjacent to \(v\). For each downward edge \(e\) at \(v\) the semigroup condition tells us that the weight \(p_e\) is a non-negative integer linear combination \(p_e = \sum \alpha_j l_{vw_j}'\), summed over the leaves below \(v\). We choose such an expression and denote by \(M_e = \prod_j z_j^{\alpha_j} \in \mathbb{C}[z_1, \ldots, z_m]\) the corresponding monomial. Then:

**Scholium.** If \((\Delta, w')\) satisfies the semigroup condition in the above theorem then the equations associated to node \(v\) are the equations that equate the monomials \(M_e\) for the different downward edges at \(v\).

If we replace each of these equations \(M_e = M'_e\) by an equation \(M_e = a_{ee'} M_{e'}\) with \(a_{ee'} \in \mathbb{C}^*\) then we obtain the same monomial curve.

**Remark 6.2.** Delorme’s Proposition 9 in [6] implies that every complete intersection semigroup arises as in Theorem 6.1. Already in the three-generator case the minimal tree defining the semigroup need not be unique.

**Example.** If \(gcd(a, b) = gcd(a, c) = gcd(c, d) = 1\) then the tree

```
    a
   / \  \
 b   c
```

satisfies the semigroup condition and leads to the complete intersection monomial curve

\[
\mathbb{C}[z_1, z_2, z_3]/(z_1^a - z_2^b, z_2^c - z_3^d) \cong \mathbb{C}[t^{bc}, t^{ac}, t^{ad}].
\]

Exchanging \(a\) with \(c\) and \(b\) with \(d\) gives a different tree for the same semigroup.

**Proof of Theorem 6.1 and Scholium.** The second part of the scholium is an easy induction once the rest is proved, replacing \(z_j \mapsto t^{l_{w_j}'}\) for \(j > 1\) by \(z_j \mapsto \lambda_j t^{l_{w_j}'}\) for suitable \(\lambda_j \in \mathbb{C}^*\). So we will just prove the theorem and first part of the scholium.

Let \(\Delta_1, \ldots, \Delta_n\) be the subtrees cut off by the bottom vertex \(w_0\) of the root edge of \(\Delta\) and let \(p_i\) be the weight on the root edge of \(\Delta_i\). Write \(\Gamma_i = \text{sg}(\Delta_i, w_0)\), \(P = p_1 \ldots p_n\) and \(P_i = P/p_i\). Then

\[
\Gamma = P_1 \Gamma_1 + \cdots + P_n \Gamma_n,
\]

the semigroup consisting of all integers of the form \(\sum P_i \gamma_i\), \(\gamma_i \in \Gamma_i\). Moreover,

\[
\mu(\Delta, w') = \sum_{i=1}^n (P_i \mu(\Delta_i, w_0) - 1) + (n - 1)P + 1.
\]

By Lemma 6.3 below, the desired results now hold for \(\Delta\) if they are true for each \(\Delta_i\). The proof is thus an induction, with the induction start being the case that \(\Delta\) consists of only a root edge and \(\text{sg}(\Delta)\) is the one-generator semigroup \(\mathbb{N}\). \(\square\)
Lemma 6.3. Suppose $\Gamma_i$ are semigroups for $i = 1, \ldots, n$, and $p_1, \ldots, p_n$ are pairwise coprime positive integers. Write $P = p_1 \ldots p_n$ and $P_i = P/p_i$. Let

$$\Gamma = P_1\Gamma_1 + \cdots + P_n\Gamma_n.$$ 

Then

(1) $2\delta(\Gamma) \leq \sum_{i=1}^n P_i(2\delta(\Gamma_i) - 1) + (n - 1)P + 1$

(2) If equality holds in (1) then $p_i \in \Gamma_i$ for $i = 1, \ldots, n$

(3) $c(\Gamma) \leq \sum_{i=1}^n P_i(c(\Gamma_i) - 1) + (n - 1)P + 1$.

(4) If $p_i \in \Gamma_i$ for $i = 1, \ldots, n$ then equality holds in (3).

(5) If each $\Gamma_i$ is symmetric then the three statements are equivalent: equality in (1), equality in (3), $p_i \in \Gamma_i$ for $i = 1, \ldots, n$.

(6) Assuming $p_i \in \Gamma_i$ for each $i$, then $\Gamma$ is symmetric resp. a complete intersection if and only if each $\Gamma_i$ is symmetric resp. complete intersection.

(7) If $p_i \in \Gamma_i$ for each $i$ then one obtains a presentation for $\Gamma$ by adjoining to the disjoint union of presentations for the $\Gamma_i$ the $n-1$ relations $w_1 = \cdots = w_n$, where $w_i$ is an expression for $p_i$ in the presentation of $\Gamma_i$.

Proof. We shall prove the case $n = 2$. The case of general $n$ follows from this case by an easy induction.

To prove (1) we count gaps in $\Gamma$. A gap $\gamma$ of $\Gamma = p_2\Gamma_1 + p_1\Gamma_2$ is either

(i) one of the $(p_1 - 1)(p_2 - 1)/2$ gaps of $p_2\mathbb{N} + p_1\mathbb{N}$,

or it is of the form $\gamma = p_2\alpha + p_1\beta$ for some $\alpha, \beta \in \mathbb{N}$. In this case we will see that either:

(ii) $\beta$ is the smallest $\beta \in \Gamma_2$ in its congruence class mod $p_2$, and $\alpha \notin \Gamma_1$, or

(iii) $0 \leq \alpha < p_1$, and $\beta \notin \Gamma_2$.

Indeed, if we can express $\gamma$ in the form $\gamma = p_2\alpha + p_1\beta$ with $\alpha, \beta \in \mathbb{N}$, then we can do so with $0 \leq \alpha < p_1$. If this expression does not satisfy condition (iii) then $\beta \in \Gamma_2$. In this case decrease $\beta$ by some multiple of $p_2$ (maybe zero) to make it the smallest $\beta \in \Gamma_2$ in its congruence class mod $p_2$, and simultaneously increase $\alpha$ by the same multiple of $p_1$ to keep $\gamma = p_2\alpha + p_1\beta$. Since $\gamma$ is a gap of $p_2\Gamma_1 + p_1\Gamma_2$, we must have $\alpha \notin \Gamma_1$, so the expression now satisfies condition (ii).

Now there are exactly $p_2\delta(\Gamma_1)$ pairs $(\beta, \alpha)$ satisfying condition (ii) and $p_1\delta(\Gamma_2)$ pairs satisfying condition (iii), so there are at most $(p_1 - 1)(p_2 - 1)/2 + p_2\delta(\Gamma_1) + p_1\delta(\Gamma_2)$ gaps of $\Gamma = p_2\Gamma_1 + p_1\Gamma_2$. This number can be written $\frac{1}{2}(p_2(2\delta(\Gamma_1) - 1) + p_1(2\delta(\Gamma_2) - 1) + p_1p_2 + 1)$, so part (1) is proven.

This proof shows that we have equality in part (1) if and only if every element $\gamma = p_2\alpha + p_1\beta$ satisfying condition (ii) or (iii) is a gap of $\Gamma$ and there is no overlap between cases (ii) and (iii). Suppose now $p_1 \notin \Gamma_1$. Then if every $p_2\alpha + p_1\beta$ satisfying (ii) is a gap of $\Gamma$, there is an overlap: $(\alpha, \beta) = (p_1, 0)$ in condition (ii) shows that $p_1p_2$ is a gap of $\Gamma$, whence $p_2 \notin \Gamma_2$, so $p_1p_2$ also has an expression with $(\alpha, \beta) = (0, p_2)$ satisfying condition (iii). Thus $p_1 \notin \Gamma_1$ implies inequality in part (1). Similarly for $p_2 \notin \Gamma_2$, so part (2) is proved.

For statement (3), we show that $i \geq 0$ added to the right hand side of the inequality of part (3) gives an element of $\Gamma$. The sum of the last two terms of $p_2c(\Gamma_1) + p_1c(\Gamma_2) + (p_1 - 1)(p_2 - 1) + i$
is in the semigroup generated by $p_1$ and $p_2$, say $p_1\alpha + p_2\beta$; so the whole expression equals

$$p_2(c(\Gamma_1) + \beta) + p_1(c(\Gamma_2) + \alpha),$$

which by definition of conductors is clearly in $\Gamma$.

For statement (4), suppose $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$, but

$$p_2c(\Gamma_1) + p_1c(\Gamma_2) + (p_1 - 1)(p_2 - 1) - 1 = p_2\lambda + p_1\pi,$$

for some $\lambda \in \Gamma_1, \pi \in \Gamma_2$.

Modulo $p_1$ this equation says $c(\Gamma_1) - 1 \equiv \lambda$, so

$$c(\Gamma_1) - 1 = \lambda + p_1t,$$

for some integer $t$.

Inserting this in the previous equation gives

$$c(\Gamma_2) - 1 = \pi + p_2(-1 - t).$$

Since one of $t$ and $-1 - t$ is $\geq 0$ and $\lambda, p_1 \in \Gamma_1$ and $\pi, p_2 \in \Gamma_2$, one gets either $c(\Gamma_1) - 1 \in \Gamma_1$ or $c(\Gamma_2) - 1 \in \Gamma_2$, a contradiction.

Part (5) is now immediate: (2) and (4) show

$$(\text{equality in (1)}) \Rightarrow (p_1 \in \Gamma_1 \text{ and } p_2 \in \Gamma_2) \Rightarrow (\text{equality in (3)}),$$

and if the $\Gamma_i$ are symmetric then $c(\Gamma) \leq 2\delta(\Gamma)$ and $c(\Gamma_i) = 2\delta(\Gamma_i)$, so equality in (3) implies equality in (1).

Part (6) is proved in [6]. (In this paper we use only that $\Gamma$ is a complete intersection if both $\Gamma_1$ and $\Gamma_2$ are; this follows from part (7).)

For part (7), let $\Gamma_1 = \langle x_1, \ldots, x_n : s_1, \ldots, s_k \rangle$ and $\Gamma_2 = \langle y_1, \ldots, y_m : r_1, \ldots, r_\ell \rangle$ be commutative semigroup presentations of $\Gamma_1$ and $\Gamma_2$ and let $p_1 = v(x_1, \ldots, x_n)$ and $p_2 = w(y_1, \ldots, y_m)$ be expressions for $p_1$ and $p_2$ in these semigroups. Suppose $p_2\gamma_1 + p_1\gamma_2 = p_2\gamma_1' + p_1\gamma_2'$ equates two elements of $\Gamma = p_2\Gamma_1 + p_1\Gamma_2$, with $\gamma_1, \gamma_1' \in \Gamma_1$ and $\gamma_2, \gamma_2' \in \Gamma_2$. Let $\gamma_1 = g_1(x_1, \ldots, x_n)$ be an expression for $\gamma_1 \in \Gamma_1$ in terms of the generators (and hence for $p_2\gamma_1$ in $p_2\Gamma_1$), and similarly $\gamma_1' = g_1'(x_1, \ldots, x_n), \gamma_2 = g_2(y_1, \ldots, y_m), \gamma_2' = g_2'(y_1, \ldots, y_m)$. Then the relation to be verified in $\Gamma = p_2\Gamma_1 + p_1\Gamma_2$ is $g_1 + g_2 = g_1' + g_2'$ (abbreviating $g_1(x_1, \ldots, x_n) = g_1$ etc.), and we must show this follows from the relations of $\Gamma_1$ and $\Gamma_2$ and the additional relation $v = w$.

With no loss of generality $\gamma_1 \geq \gamma_1'$ in $\mathbb{N}$. Then, working in $\mathbb{N}$, we have $p_2(\gamma_1 - \gamma_1') = p_1(\gamma_1' - \gamma_2)$, so $\gamma_1 - \gamma_1' = sp_1$ and $\gamma_1' - \gamma_2 = sp_2$ for some $s \in \mathbb{N}$. In particular, the equations $g_1 = sv + g_1'$ and $g_2' = sw + g_2$ hold in $\Gamma_1$ and $\Gamma_2$, so they must follow from the relations of these semigroups. Thus, using the additional relation $v = w$, we deduce $g_1 + g_2 = sv + g_1' + g_2 = sw + g_1' + g_2 = g_1' + g_2$, as desired. $\square$

6.1. Normal form monomials. Suppose now that $(\Delta, w')$ satisfies the semigroup condition and put $\Gamma = \text{sg}(\Delta)$. We wish to describe a monomial basis for the corresponding complete intersection curve $\mathbb{C}[z_1, \ldots, z_m]/(\text{relations})$. That is, we want “normal form” monomials in $z_1, \ldots, z_m$ so that each $t^\gamma$ with $\gamma \in \Gamma$ is the image of exactly one monomial under the map $\mathbb{C}[z_1, \ldots, z_m] \to \mathbb{C}[t^\Gamma]$ given by $z_j \mapsto t^{w'(z_j)}$. We will do this by systematically trying to eliminate variables with small index.

We assume that the tree $\Delta$ is drawn so that the indices $i = 1, \ldots, m$ of the leaves increase from left to right. For any node $v$ and outward edge $e$ at $v$ let $\Delta_{ve}$ be the subtree below $v$ with root vertex $v$ and root edge $e$.

If $M$ is a monomial, let $M_{ve}$ be the submonomial of $M$ determined by the variables corresponding to leaves of $\Delta_{ve}$. This monomial represents $t^\alpha \in \mathbb{C}[t^{\text{sg}(\Delta_{ve})}]$.
for some $\alpha$. We will say $M$ is in normal form if for every $v$ and $e$ as above so that $e$ is not the rightmost edge at $v$, $\alpha - p_e \notin \sg(\Delta_{ve})$.

If $M$ is not in normal form at some $(v, e)$ then we could replace $M_{ve}$ in $M$ by $M'_{ve}$ where $e'$ is the rightmost edge at $v$, $M'_{ve}$ is a monomial representing $t^{\alpha - p_e} \in \mathbb{C}[t^{\sg(\Delta_{ve})}]$ and $M'_{e'}$ is a monomial representing $t^{p_{e'}} \in \mathbb{C}[t^{\sg(\Delta_{ve'})}]$. Since $t^{p_e} \in \mathbb{C}[t^{\sg(\Delta_{ve})}]$ and $t^{p_{e'}} \in \mathbb{C}[t^{\sg(\Delta_{ve'})}]$ become equal in $\mathbb{C}[t^{\sg(\Delta)}]$, this does not change the value of $M$. It is easy to see this process must eventually stop. A simple induction shows that it yields a unique normal form for $M$. Normal form monomials thus provide the desired monomial basis of $\mathbb{C}[z_1, \ldots, z_m]/(relations)$.

The following example will be important in the next section.

**Example 4.** Let

$$\Delta = \begin{array}{c}
\quad \\
p_1 \\
\quad \\
\quad \\
\quad \\
\quad \\
\quad \\
\quad \\
\quad \\
\quad \\
\end{array}$$

so $\Gamma$ is the semigroup generated by the $P_i = P/p_i$. The monomial curve

$$(t^{P_1}, t^{P_2}, \ldots, t^{P_n})$$

is the complete intersection curve singularity defined by the equations

$$z_1^{P_1} - z_n^{P_n} = 0, \quad i = 1, \ldots, n - 1.$$

The conductor $c(\Gamma)$ is

$$P \left( n - 1 - \sum (1/p_i) \right) + 1.$$

The monomial basis described above is

$$\{z_1^{\alpha_1} \ldots z_n^{\alpha_n} : \alpha_i < p_i \text{ for all } i = 1, \ldots, n - 1\}.$$

More generally, applied to a tree of the form

$$\Delta = \begin{array}{c}
p_1 \\
p_{k_1} \\
\quad \\
p_{k_2} \\
\quad \\
p_{k_{r+1}} \\
\quad \\
p_n \\
\quad \\
\quad \\
\quad \\
\quad \\
\end{array}$$

which satisfies the semigroup condition, the above procedure will again give the monomial basis

$$\{z_1^{\alpha_1} \ldots z_n^{\alpha_n} : \alpha_i < p_i \text{ for all } i = 1, \ldots, n - 1\}.$$

(However, with a different ordering of the variables the monomial basis for this example can be considerably more complicated.)
7. Geometric genus and Theorem 2

In this section we will prove Theorem 2 by computing geometric genus (see Theorem 3.3).

Let \((X,o)\) be a germ of a normal surface singularity, with analytic local ring \(O\). Consider a good resolution \(\pi: (Y,E) \to (X,o)\), i.e., the exceptional fiber \(E = \bigcup E_i\) is a union of smooth curves intersecting transversely, no three through a point. By local duality, one may compute the geometric genus in two ways:

\[
p_g(X) = \dim H^1(O_Y) = \dim H^0(U, K_U)/H^0(Y, K_Y),
\]

where \(U = X - \{o\} = Y - E\), and \(K\) denotes canonical line bundle (or its sheaf of sections).

If \((X,o)\) is Gorenstein, let \(\omega\) be a nowhere-0 holomorphic two-form on \(U\). Define the canonical ideal \(J\) of \(O\) by

\[
J = \{ f \in O : f\omega \text{ is regular on } Y \}.
\]

Then clearly

\[
p_g(X) = \dim O/J.
\]

Let \(E_\alpha, \alpha = 1, \ldots, t\) be those exceptional curves which either have positive genus, or intersect at least three other curves. Let \(G\) be the union of the remaining curves (the “strings” in the resolution). The blowing-down \(Y \to Y'\) of \(G\) gives a space with only cyclic quotient singularities (if \(Y\) is the minimal good resolution then \(Y'\) is the “log-canonical resolution”); since these singularities are rational, regular forms in a punctured neighborhood automatically extend regularly on a resolution. Therefore, \(f \in J\) if and only if \(f\omega\) extends regularly over the \(t\) particular curves \(E_\alpha\). Let \(\nu_\alpha\) be the valuation on \(O\) given by order of vanishing along \(E_\alpha\), and let \(k_\alpha + 1\) denote the order of the pole of \(\omega\) along that curve. We conclude that

\[
J = \{ f \in O : \nu_\alpha(f) \geq k_\alpha + 1, \alpha = 1, 2, \ldots, t \}.
\]

Proposition 7.1. Let \((X,o)\) be the germ of a Gorenstein surface singularity, whose link is a rational homology sphere. Let \((Y,E) \to (X,o)\) be the minimal good resolution, and let \(E_1, \ldots, E_i\) be the exceptional curves of valency \(\geq 3\). Let \(k_\alpha\) be the coefficient of \(E_\alpha\) in the divisor \(- (K + E)\), and \(\nu_\alpha\) the corresponding valuation of the local ring \(O\) of \(X\). Then the geometric genus of \(X\) is the colength of the ideal

\[
J = \{ f \in O : \nu_\alpha(f) \geq k_\alpha, \alpha = 1, \ldots, t \}.
\]

Proof. By the preceding discussion, the statement to be proved is

\[
H^0(Y, K_Y) = H^0(Y - G, K_Y + E).
\]

We will do this in two steps:

\[
H^0(Y, K_Y) = H^0(Y, K_Y + E) = H^0(Y - G, K_Y + E).
\]

Since the link of \(X\) is a \(\mathbb{Q}\)-homology-sphere, the exceptional curve \(E\) is the transverse union of smooth rational curves \(E_i\), no three through a point, with contractible dual graph. It follows that \(h^1(O_E) = 0\). (Proof: write \(E = E_1 + F\), where \(E_1\) is a component of \(E\) that meets the rest \(F\) of \(E\) in a single point; the surjection \(O_E \to O_F\) has kernel \(O_{E_1}(-F) = O(-1)\), so the claim follows by induction on the number of components of \(E\).)
Denote $K_Y \otimes \mathcal{O}_E(E)$ by $K_E$ (called the dualizing sheaf in [1] section II.1). Serre duality implies that, for any line bundle $L$ on $E$, $H^1(E, L)$ is dual to $H^0(E, L^* \otimes K_E)$ (e.g., [1], Theorem II(6.1)). Taking $L$ trivial we see $h^0(K_E) = 0$. The adjunction sequence $0 \to K_Y \to K_Y + E \to K_E \to 0$ (called “residue sequence” in [1] section II.1) now gives $0 \to H^0(K_Y) \to H^0(K_Y + E) \to H^0(K_E) = 0$, proving the first equality $H^0(K_Y) = H^0(K_Y + E)$.

The second equality $H^0(Y, K_Y + E) = H^0(Y - G, K_Y + E)$ holds generally, without the condition on the link. In fact, if $G$ is any union of components of $E$ and $L$ any divisor supported on $E$ then it is easy to see that $H^0(Y, L) = H^0(Y - G, L)$ so long as $L \cdot G_i \leq 0$ for each component of $G$ (for a stronger statement see [8]), so we must just show that that $(K + E) \cdot G_i \leq 0$ for all $i$. But $G_i$ is a smooth rational curve, so $(K + E) \cdot G_i$ equals $-2$ plus the number of intersections of $G_i$ with the other curves of $E$. This result is $-1$ if $G_i$ is an end curve of the graph, or $0$ otherwise. In either case, the condition is fulfilled, and our result follows. \hfill \Box

While the $k_i$ are determined from the resolution graph (see Proposition 4.1), in some cases they can be computed directly from the equations defining $\mathcal{O}$. 

**Proposition 7.2.** Let 
\[ \mathbb{C}[z_1, \ldots, z_s]/(f_1, \cdots, f_{s-2}) \]
define an isolated complete intersection surface singularity at the origin. For an exceptional curve $E_1$ in a resolution, with valuation $\nu = \nu_1$, consider the filtration defined by $I_n = \{ f : \nu(f) \geq n \}$. Assume that the associated graded of this filtration is a complete intersection integral domain, with the $z_i$ inducing homogeneous generators, and defined by the $\nu$-leading forms $f_j$, $j = 1, \cdots, s - 2$. Then the invariant $k_1$ is computed as
\[ k_1 = \sum_{j=1}^{s-2} \nu(f_j) - \sum_{i=1}^{s} \nu(z_i). \]

**Proof.** We may interpret
\[ \omega = dz_1 \wedge \cdots \wedge dz_s / df_1 \wedge \cdots \wedge df_{s-2}. \]
On the associated graded, this gives a two-form of total weight
\[ \sum \nu(z_i) - \Sigma \nu(f_j). \]
In terms of local coordinates in a neighborhood of a general point of $E_1$, one finds the order of the pole of $\omega$ is one more than the weight, as desired. \hfill \Box

If our singularity is a complete intersection of splice type and $E_1$ corresponds to a node $v$ of the splice diagram, then, in the terminology of the preceding section, $\nu(z_i)$ is the $\nu$-weight of $z_i$, so $\nu(z_i)$ is the product of splice diagram weights adjacent to the path from node $v$ to leaf $i$. It is easy to see that the formula of the above proposition is then equivalent to that of Proposition 4.1.

**Example 5.** The last two propositions give a well-known result for a weighted homogeneous complete intersection: the geometric genus is the sum of the dimensions of the graded pieces of weight less than or equal to $k_1$ above. In particular, let $V(p_1, \ldots, p_n)$ (with $p_i$ pairwise relatively prime) be a Brieskorn-Pham complete intersection, defined by
\[ z_1^{p_1} + a_1 z_{n-1}^{p_n} + b_1 z_n^{p_n} = 0 \]
\[ z_2^{p_2} + a_2 z_{n-1}^{p_n} + b_2 z_n^{p_n} = 0 \]
\[
\cdots z_n^{P_n-2} + a_{n-2}z_n^{P_{n-1}} + b_{n-2}z_n^{P_n} = 0.
\]

Let \( P = p_1 \cdots p_n, P_i = P/p_i. \) Then
\[
k_i = (n-2)P - \Sigma P_i = P(n-2 - \Sigma(1/p_i)).
\]

Using the monomial basis \( z_1^{i_1} \cdots z_n^{i_n} \) with \( i_k < p_k, k = 1, \cdots, n - 2, \) one computes
\[
p_g(V(p_1, \ldots, p_n)) = \# \{(i_1, i_2, \cdots, i_n) \in (\mathbb{Z}_{\geq 0})^n : \sum_{k=1}^n (i_k + 1)/p_k < n - 2; i_k < p_k, k = 1, \cdots, n - 2 \).
\]

To extend this calculation to singularities of splice type for more complicated splice diagrams we need to be able to find a monomial basis for the filtration defined by any node of the splice diagram.

Suppose we have a complete intersection \((X, p)\) of splice type corresponding to a splice diagram \(\Delta\). Let \(\nu\) be the valuation associated to the node \(v\) of \(\Delta\). Let the edges around \(v\) be \(e_1, \ldots, e_n\) with weights \(d_{ve_i} = p_i, i = 1, \ldots, n\) at \(v\). For each node \(v'\) of \(\Delta\) the equations have the form \(\sum a_{e'} M_{e', e} = 0\), sum over the edges \(e'\) at \(v'\), where \(M_{e', e}\) is an admissible monomial at \(v'\) and \(a_{e'} \in \mathbb{C}\). If \(v' \neq v\) and \(e'\) is the edge on the path from \(v'\) to \(v\) we will call \(M_{v', e'}\) a near monomial at \(v'\) for \(v\). Thus there is one near monomial for \(v\) associated to each node other than \(v\).

**Theorem 7.3.** The associated graded ring \(R\) of \((X, p)\) with respect to the filtration associated to \(\nu\) is a reduced and irreducible complete intersection, defined by the same equations as \((X, p)\) but with the coefficients of all near monomials for \(v\) set to zero (so only the equations associated to the node \(v\) remain unchanged). Its normalization is the Brieskorn-Pham complete intersection \(V(p_1, \ldots, p_n)\).

We will need a specific basis of the graded ring \(R\).

**Proposition 7.4.** Choose an edge \(e\) at \(v\) and picture the edge \(e\) as horizontal, with \(v\) on the left. Cut the edge \(e\) at its midpoint and use this midpoint as root of the resulting trees \(\Delta^L\) on the left and \(\Delta^R\) on the right (so these are rooted trees and \(\Delta^L\) contains \(v\)). Let \(M^L_e\) and \(M^R_e\) be monomial bases for the monomial curves of \(\Delta^L\) and \(\Delta^R\), constructed as in subsection 6.1. Then the set of monomials \(M^L_e \cup M^R_e\) forms a \(\mathbb{C}\)-basis of the associated graded ring \(R\). The integer \(k_v\) is given by

\[
k_v = d_{ve}(C^L - 1) + d_{ve}(C^R - 1) = p_j(C^L - 1) + C_j(C^R - 1) \text{ if } e = e_j
\]

where \(C^L\) and \(C^R\) are the conductors of the semigroups \(\text{sg}(\Delta^L)\) and \(\text{sg}(\Delta^R)\).

**Proof of Theorem 7.3 and Proposition 7.4.** Let \(v'\) be the valuation associated to \(v'\). If \(z_w\) is the variable associated to a leaf \(w\) then one checks easily that

\[
\frac{\nu(z_w)}{\nu'(z_w)} = \frac{\ell^{D_{e_1} \cdots D_{e_k}}_{v', w}}{d_{v', w}}
\]

where:

- \(\ell^{D_{e_1} \cdots D_{e_k}}_{v', w}\) is, as usual, the product of weights adjacent to the path from \(v'\) to \(v\), omitting weights at \(v'\);
- \(d_{v', w}\) is the weight at \(v'\) on the first edge \(e' = e'_1\) on the path from \(v'\) to \(v\).
• \( e'_1, \ldots, e'_k \) are the edges that are on the path from \( v' \) to \( v \) but not on the path from \( w \) to \( v \);
• for any edge \( e \), \( D_e \) is the product of the edge weights on \( e \) divided by the product of edge weights directly adjacent to \( e \) (so \( D_e > 1 \) by the edge determinant condition).

Thus \( \nu(z_w)/\nu'(z_w) \) takes its minimum value (namely \( \ell(v',v)/d(v',v') \)) if and only if \( w \) is beyond \( v' \) from the point of view of \( v \). It follows that the admissible monomials at \( v' \) all have the same \( \nu \)-weight except for the near monomial for \( v \), which has higher \( \nu \)-weight. Hence, the equations which define the associated graded ring \( R \) are obtained by setting coefficients of near monomials equal to zero.

For convenience of notation we will take \( e = e_n \) for this proof. We may assume (see Section 2) that the equations associated to the node \( v \) are

\[
M_{e_1} + a_1 M_{e_{n-1}} + b_1 M_{e_n} = 0 \\
\vdots \\
M_{e_{n-1}} + a_{n-2} M_{e_{n-1}} + b_{n-2} M_{e_n} = 0
\]

For each \( i = 1, \ldots, n \), let \( \Delta_i \) be the tree cut off away from \( v \) at the midpoint of \( e_i \) (so \( \Delta_i = \Delta_i^R \)). By the Scholium to Theorem 6.1, the equations for the associated graded \( R \) that correspond to nodes in \( \Delta_i \) give a complete intersection description of the monomial curve \( \mathbb{C}[\text{sg}(\Delta_i)] \). Let \( \phi_i : \mathbb{C}[z_w, w \text{ a leaf of } \Delta_i] \to \mathbb{C}[\text{sg}(\Delta_i)] \) be the corresponding homomorphism. Then \( \phi_i(M_{e_n}) = c_i X^{p_i} \) for some \( c_i \in \mathbb{C}^* \). Together these homomorphisms \( \phi_i \) give a homomorphism \( \phi \) of \( R \) to the Brieskorn-Pham complete intersection defined by the equations

\[
\begin{align*}
c_1 X^{p_1} + c_{n-1} a_1 X^{p_{n-1}} + c_n b_1 X_n^{p_n} &= 0 \\
& \quad \vdots \\
c_{1} X_{1}^{p_{1}} + c_{n-1} a_{n-2} X_{n-1}^{p_{n-1}} + c_n b_{n-2} X_n^{p_n} &= 0
\end{align*}
\] (6)

Let \( z_1, \ldots, z_k \) be the variables corresponding to nodes of \( \Delta \) in \( \Delta^L_{e} \) and \( z_{k+1}, \ldots, z_N \) the remaining variables, corresponding to nodes in \( \Delta^R_{e} \). The graded equations corresponding to nodes in \( \Delta^L_{e} \) are equations for the complete intersection curve defined by \( \Delta^L_{e} \) except for additional terms \( b_i M_{e_n} \) (in the equations corresponding to node \( v \)). The procedure of subsection 6.1 to put a monomials in normal form will therefore change a monomial \( M \) in the variables \( z_1, \ldots, z_k \) into a linear combination of monomials of the form \( M'M_{e_n}^\alpha \), \( \alpha \geq 0 \), with \( M' \in \Delta^L_{e} \). Thus, given any monomial in \( z_1, \ldots, z_N \), we first apply the graded equations corresponding to nodes in \( \Delta^L_{e} \) to put anything involving \( z_1, \ldots, z_k \) in \( \Delta^L_{e} \)-normal form (at the expense of adding factors \( M_{e_n} \)), and then apply the graded equations corresponding to nodes in \( \Delta^R_{e} \) to put anything involving \( z_{k+1}, \ldots, z_N \) into \( \Delta^R_{e} \)-normal form. It follows that the set \( \Delta^L_{e} \Delta^R_{e} \) is a \( \mathbb{C} \) spanning set for the graded ring \( R \). On the other hand, one can check that the set

\[
\{ \phi(M_1 M_2) : M_1 \in \Delta^L_{e}, M_2 \in \Delta^R_{e} \} \subset \mathbb{C}[X_1, \ldots, X_n]/(\text{relations (6)})
\]

is linearly independent (we will not give a detailed proof of this, since it is immediate in the case below to which we apply this proposition). Hence \( \Delta^L_{e} \Delta^R_{e} \) is a monomial basis for \( R \). Moreover, since \( \phi \) is birational, \( \phi \) is the normalization of \( R \). Finally, the calculation of \( k_v \) is straightforward, using either Proposition 7.2 or Proposition 4.1. \( \square \)
Note that the monomial basis given by the above proposition depends on the choice of edge and also on the ordering of the variables. Although the proposition gives the same monomial basis for the valuations corresponding to the two ends of the edge, if we take a different node we will have to take a different edge and will in general get a different monomial basis. However, to apply this proposition to compute the geometric genus of \((X, p)\) we shall need the same monomial basis for all the valuations. This turns out to be possible for the splice diagram \(\Delta\).

We number the nodes and edges of this diagram \(v_0, \ldots, v_r\) and \(e_1, \ldots, e_r\) from left to right. The valuation for node \(v_i\) will be denoted \(\nu_i\).

For the edge \(e = e_i\) joining nodes \(v_{i-1}\) and \(v_i\) we divide the variables \(z_1, \ldots, z_N\) into two groups, ordered as follows:

\[
\begin{align*}
&z_{k_i}, z_{k_i-1}, \ldots, z_1, \\
&z_{k_i+1}, z_{k_i+2}, \ldots, z_N
\end{align*}
\]

We apply the above proposition for this particular edge \(e\). Example 4 gives the monomial bases

\[
\begin{align*}
\mathcal{M}_L &= \{z_1^{\alpha_1} \cdots z_{k_i}^{\alpha_{k_i}} : 0 \leq \alpha_i < p_i \text{ for } i = 2, \ldots, k_i\} \\
\mathcal{M}_R &= \{z_{k_i+1}^{\alpha_{k_i+1}} \cdots z_N^{\alpha_N} : 0 \leq \alpha_i < p_i \text{ for } i = k_i+1, \ldots, N-1\}
\end{align*}
\]

for the two semigroups in question, so we get:

**Lemma 7.5.** For each valuation \(\nu_i\) of the above \(\Delta\),

\[
\mathcal{M} := \{z_1^{\alpha_1} \cdots z_N^{\alpha_N} : 0 \leq \alpha_i < p_i \text{ for } i = 2, \ldots, N-1\}
\]

is a monomial basis for the associated graded ring \(R\). 

We continue to consider the edge \(e = e_i\) of \(\Delta\) with left end node \(v = v_{i-1}\). We can consider \(\Delta_L\) and \(\Delta_R\) also as splice diagrams, and then \(\Delta\) is the result of splicing them at their root leaves.

**Theorem 7.6.** The geometric genus \(p_g(\Delta)\) of the splice type singularity determined by \(\Delta\) is given inductively by

\[
(1/4)C^L C^R + p_g(\Delta_L) + p_g(\Delta_R)
\]

where \(C^L, C^R\) are the conductors of the semigroups \(sg(\Delta_L, w')\) and \(sg(\Delta_R, w')\).

**Corollary 7.7.** The Casson Invariant Conjecture holds for the splice type singularity determined by the above splice diagram \(\Delta\).

**Proof of Corollary.** By Theorem 3.3, the corollary follows by induction as soon as we know that the geometric genus satisfies an appropriate formula. The formula of Theorem 7.6 is the right one since \(C^L\) and \(C^R\) are the Milnor numbers of the knots corresponding to the root leaves of \(\Delta_L\) and \(\Delta_R\) (Theorem 2.2). 

\(\square\)
Proof of Theorem 7.6. The canonical ideal of the singularity \((X, p)\) consists of those \(f\) for which \(\nu_j(f) \geq k_{v_j}\) for \(j = 0, \ldots, r\). Using the linearly independent monomials of the above lemma, the geometric genus thus equals the number of elements of
\[
\mathcal{M} = \{z_1^{\alpha_1} \cdots z_N^{\alpha_N} : 0 \leq \alpha_i < p_i \text{ for } i = 2, \ldots, N - 1\}
\]
satisfying
\[
\nu_i(M) < k_{v_i} \quad \text{for some } i = 0, \ldots, r.
\]
We will call this condition “condition \(K(v_i)\)” so we want to count the \(M \in \mathcal{M}\) for which condition \(K(v)\) holds for some node \(v\).

Let \(e = e_i\). For a monomial \(M = z_1^{\alpha_1} \cdots z_N^{\alpha_N}\), write \(M = M_L M_R\) with \(M_L = z_1^{\alpha_1} \cdots m_{k_1}\) and \(M_R = z_{k_1+1}^{\alpha_{k_1+1}} \cdots z_N^{\alpha_N}\). The monomial \(M\) is in \(\mathcal{M}\) if and only if \(M_L\) and \(M_R\) are normal form monomials for the semigroups \(\text{sg}(\Delta_e^L)\) and \(\text{sg}(\Delta_e^R)\).

Denote the nodes at the left and right end of \(e = e_i\) by \(v = v_{i-1}\) and \(v' = v_i\) and the associated valuations by \(\nu = \nu_{i-1}\) and \(\nu' = \nu_i\). Denote
\[
\ell_e(M_L) := \sum_{j=1}^{k_i} \alpha_j \ell_{w' w_j}, \quad \ell_e(M_R) := \sum_{j=k_i+1}^{N} \alpha_j \ell_{w' w_j}
\]
where \(w'\) is the root vertex of \(\Delta_e^L\) or \(\Delta_e^R\) and \(\ell_{w' w_j}\) is computed in \(\Delta_e^L\) or \(\Delta_e^R\). (Thus \(\ell_e(M_L)\) and \(\ell_e(M_R)\) are the values in the semigroups \(\text{sg}(\Delta_e^L)\) and \(\text{sg}(\Delta_e^R)\) corresponding to the monomials \(M_L\) and \(M_R\).)

By Proposition 7.4 condition \(K(v_{i-1})\) can thus be written
\[
d_{w e} (\ell_e(M_L) - C^L + 1) + (d_v/d_{w e}) (\ell_e(M_R) - C^R + 1) < 0
\]
By symmetry, condition \(K(v_i)\) can be written
\[
(d_{v'}/d_{w' e}) (\ell_e(M_L) - C^L + 1) + d_{w' e} (\ell_e(M_R) - C^R + 1) < 0.
\]
Denote
\[
X_i := \ell_e(M_L) - C^L + 1 \quad Y_i := \ell_e(M_R) - C^R + 1,
\]
so (8) and (9) can be written
\[
K(v_{i-1}) : \quad d_{w e} X_i + (d_v/d_{w e}) Y_i < 0,
\]
\[
K(v_i) : \quad (d_{v'}/d_{w' e}) X_i + d_{w' e} Y_i < 0.
\]
Note that \(X_i \neq 0\) since \(\ell_e(M_L)\) is in the semigroup \(\text{sg}(\Delta_e^L)\) with conductor \(C^L\). Similarly \(Y_i \neq 0\). We will count the monomials \(M \in \mathcal{M}\) that satisfy condition \(K(v)\) for some node \(v\) by subdividing into the following cases.

1. \(X_i < 0\) and \(Y_i < 0\) (so \(K(v_{i-1})\) and \(K(v_i)\) hold),
2. \(Y_i > 0\) and \(K(v_j)\) holds for some \(j \leq i - 1\),
3. \(X_i > 0\) and \(K(v_j)\) holds for some \(j \geq i\),
4. \(Y_i > 0\) and \(K(v_j)\) holds for some \(j \geq i\) and fails for all \(j \leq i - 1\).
5. \(X_i > 0\) and \(K(v_j)\) holds for some \(j \leq i - 1\) and fails for all \(j \geq i\).

These cases cover all possibilities. We shall show that cases (4) and (5) are empty and that Cases (1), (2), (3) are mutually exclusive and lead to the three terms on the right in the theorem.

1. The number of monomials \(M_L\) in normal form with \(\ell_e(M_L) < C^L - 1\) is the number of elements bounded by \(C^L\) in the semigroup \(\text{sg}(\Delta_e^L)\). This is exactly
$C^L/2$. Similarly for $M_R$, so the set of $M \in M$ with both $\ell_c(M_L) - C^L + 1 < 0$ and $\ell_c(M_R) - C^R + 1 < 0$ contributes the $(1/4)C^LC^R$ of the theorem.

(2) The inequality $\ell_c(M_R) - C^R + 1 > 0$ says $\ell_c(M_R) \geq C^R$, so there exists a unique monomial $M_R$ in normal form with such a value of $\ell_c(M_R)$. That is, if we put $\alpha = \ell_c(M_R) - C^R$ then there is no constraint on $\alpha \geq 0$ for a corresponding $M_R$ to exist. Consider the monomials $M_L M_R$ and $M_L z^\alpha$, which are normal form monomials for the splice diagrams $\Delta$ and $\Delta^L_e$ respectively. A simple calculation, which we omit, shows that $M_L M_R$ satisfies condition $K(v_j)$ for $j \leq i - 1$ if and only if $M_L z^\alpha$ satisfies $K(v_j)$ for $\Delta^L_e$. Thus the monomials $M = M_L M_R$ satisfying (2) are in one-one correspondence with the monomials that count $\ell_c(\Delta^L_e)$.

(3) By symmetry, these monomials count $p_y(\Delta^L_e)$.

(4) One calculates that

$$Y_i = (d_{v_i}/q_i q_{i+1}) Y_{i+1} + (d_{v_i}/q_i) \sum_{j=k_{i+1}}^{k_{i+1}} \frac{1}{p_j} (\alpha_j + 1 - p_j)$$

(we are using the explicit weights $d_{v_i,e} = q'_i$ etc. from the picture of $\Delta$). Since $\alpha_j < p_j$ for $j = k_i + 1, \ldots, k_{i+1}$, the sum on the right is non-positive so $Y_i > 0$ implies $Y_{i+1} > 0$. Thus, if we are in case (4) we can, by increasing $i$ if necessary, assume that $Y_i > 0$ and $K(v_j)$ holds for $j = i$ and fails for $j = i - 1$. By (10) we then have

$$d_{v_i}/d_{v_i}^{2e} > -X_i/Y_i, \quad d_{v_i}^{2e}/d_{v_i} < -X_i/Y_i.$$ 

Thus $d_{v_i}/d_{v_i}^{2e} > d_{v_i}/d_{v_i}$, whence $d_{v_i} d_{v_i}/(d_{v_i} d_{v_i}) > d_{v_i} d_{v_i}$, contradiction the edge determinant condition. Thus case (4) cannot happen, and by symmetry the same holds for case (5).

It remains to show that cases (2) and (3) are mutually exclusive (Case (1) is clearly disjoint from (2) and (3)). But if both (2) and (3) hold then $K(v_i)$ must fail. Since $K(v_j)$ holds for some $j > i$ and $Y_i > 0$, the same argument as in (4) leads to a contradiction. \hfill \Box

8. The semigroup condition

Let $(X, o)$ be a normal surface singularity whose link $\Sigma$ is an integral homology sphere. Each leaf of the splice (or resolution) diagram gives a knot in $\Sigma$, unique up to isotopy. A key point in the proof in [25], that splice diagram equations give integral homology sphere links, is to show that the variable $z_i$ associated to a leaf cuts out the corresponding knot in $\Sigma$. In other words, the curve $C_i$ given by $z_i = 0$ is irreducible, and its proper transform $D_i$ on the minimal good resolution is smooth and intersects transversely the exceptional curve corresponding to the leaf of the splice diagram. We show that the existence of such functions implies the semigroup condition on the splice diagram.

**Theorem 8.1.** Let $(X, o)$ be a normal surface singularity whose link $\Sigma$ is an integral homology sphere. Assume that for each of the $t$ leaves $w_i$ of the splice diagram $\Delta$ of $\Sigma$, there is a function $z_i$ inducing the end knot as above. Then

(1) $\Delta$ satisfies the semigroup condition

(2) $X$ is a complete intersection of embedding dimension $\leq t$

(3) $z_1, \ldots, z_t$ generate the maximal ideal of the local ring of $X$ at $o$, and $X$ is a complete intersection of splice type with respect to these generators (up to higher weight deformation).
Proof. Let \((Y, E) \to (X, o)\) be the minimal good resolution, \(z = z_1\) a function as above, \(C \subset X\) the irreducible Cartier divisor defined by \(z = 0\), \(D \subset Y\) its proper transform, and \(E_1 \subset Y\) the exceptional curve (which intersects \(D\) in one point) corresponding to the leaf of the splice diagram.

Let \(V\) be the value semigroup of \(C\). The orders of vanishing of the functions \(z_2, \ldots, z_t\) at \(D \cap E_1\) generate a subsemigroup \(\Gamma \subset V\) which we can compute from \(\Delta\) as follows. For each exceptional curve \(E_i\), let \(a_{ij}\) be the order of vanishing of \(z_j\) on \(E_i\), so, as a divisor, \(z_j^{-1}(0) = \sum a_{ij} E_i + D_j\). The equations \(z_j^{-1}(0) \cdot E_k = 0\) imply that \(a_{ij}\) is the \(ij\)-entry of the matrix \((-E_i \cdot E_j)^{-1}\); so by Theorem 9.1 of the Appendix (Section 9, \(a_{ij} = \ell_{ij}\)). Thus \(\Gamma\) is the semigroup generated by \(\ell_{1j}, j \geq 2\).

Theorem 6.1 implies \(2\delta(\Gamma) \leq \mu(\Delta, w_1)\), where \(\mu(\Delta, w_1)\) is described there and \(\delta(\Gamma)\) denotes the number of gaps of \(\Gamma\). But, by Theorem 2.2, \(\mu(\Delta, w_1)\) is also equal to the \(\mu\)-invariant \(\mu(C)\) of the curve \(C\). Now \(\mu(C) = 2\delta(V)\) (since we do not make a priori that the curve is Gorenstein, we must appeal to Buchweitz and Greuel [2] for this). Since the inclusion \(\Gamma \subset V\) implies \(\delta(V) \leq \delta(\Gamma)\), we conclude that \(2\delta(V) = 2\delta(\Gamma) = \mu(\Delta, w_1)\). Thus \(\Gamma = V\), and, by Theorem 6.1 again, \(\Gamma = V\) is a complete intersection semigroup. This implies that \(C\) is a positive weight deformation of the monomial curve \(\mathbb{C}[t^\gamma : \gamma \in \Gamma]\) (e.g., Teissier’s appendix to [30] or [26]) and in particular is itself a complete intersection (with maximal ideal generated by the images of \(z_2, \ldots, z_t\)). It follows that \((X, o)\) is a complete intersection (with maximal ideal generated by \(z_1, \ldots, z_n\)). Finally, repeating the argument at every leaf gives all the semigroup conditions.

It remains to show that, using the functions \(z_1, \ldots, z_t\) above, we can find equations which are equisingular deformations (even higher weight deformations) of the standard splice equations. This will proceed as follows: for each node \(v\) of valency \(\delta = \delta_v\), we will write down appropriate monomials in the \(z_i\) which have the same weight at the node (i.e., order of vanishing along the corresponding exceptional curve), and conclude there are \(\delta - 2\) independent linear dependence relations among these monomials, mod higher weight terms.

Let \(E_v\) be the exceptional curve corresponding to the node \(v\) and let \(E_1, \ldots, E_\delta\) be the exceptional curves with intersect \(E_v\), corresponding to the edges \(e_1, \ldots, e_\delta\) at \(v\). Choose a monomial \(M_i\) of weight \(d_i\) associated to each edge \(e_i\) at \(v\) (their existence is guaranteed by the semigroup condition). On the exceptional curve \(E_v\) these monomials all vanish to order \(d_i\). If we go to an adjacent node \(v'\) of the splice diagram, as in

```
   p1  v  p2  q2  v'  q3  ...
   p2  p3  q2  q3  q4  ...
   ...
```

then the order of vanishing of \(M_i\) is \(p_1 \ldots p_{\delta-1} q_1 \ldots q_{\delta'-1}\) for \(i \neq \delta\) and the order of vanishing of \(M_\delta\) is \(p_3 q_0\). In particular, at the exceptional curve corresponding to \(v'\), \(M_\delta\) vanishes to order \(D\) more than the other \(M_i\)'s, where \(D\) is the edge determinant of edge \(e_\delta\). Now in the (unreduced) maximal splice diagram (see the Appendix; Section 9) we have a node for every exceptional curve and all edge determinants are 1. Thus we see that on each exceptional curve \(E_i\) that intersects \(E_v\), the \(M_j\) with \(j \neq i\) vanish to a common order and \(M_i\) vanishes to one higher order. Thus, if we fix one of the neighboring exceptional curves, say \(E_\delta\), then each ratio \(M_i/M_\delta\) for \(i \neq \delta\) gives a function on \(E_v\) that has a pole of order 1 at the point
Ev ∩ Eδ, a simple zero at the point of intersection Ev ∩ Ei, and no other poles or zeros. It follows that there are δ − 2 linearly independent relations among the Mi up to higher order at Ev, as desired.

This gives us a collection of higher weight perturbations of equations of splice type and they are the complete intersection description of (X, o) since they give the appropriate complete intersection curves when intersected with zj = 0.

It is a Riemann-Roch problem to determine if a Gorenstein singularity with homology sphere link has functions z with the properties described above. An equivalent formulation is as follows: Let F1 be the effective exceptional cycle on the minimal good resolution Y so that it dots to 0 with every exceptional curve, save E1, with which it dots to -1. Then O(−F1) should be a base-point free line bundle. (In such a case, a generic section z will have the desired property). However, it is not even known if there is any function at all giving an irreducible divisor on X; this is certainly not the case for a general hypersurface singularity [13].

We give an application of the above theorem. Recall that we showed in Section 5 that if a surface singularity of the form zn = g(x, y) has homology sphere link, then there is a splice type singularity with the same topology (and this singularity is analytically equivalent to one given by an equation of the form zn = f(x, y)). This left open the question whether the original singularity zn = g(x, y) is an equisingular deformation of the splice type singularity zn = f(x, y).

Corollary 8.2. Any surface singularity with homology sphere link given by an equation zn = g(x, y) is a higher weight deformation of a splice type singularity.

Proof. We just sketch the proof. If the splice diagram for the plane curve g(x, y) = 0 is

\[
p_1 \quad 1 \quad p_2 \quad 1 \ldots \quad p_k \quad 1 \quad q_1 \\
\quad q_1 \quad q_2 \quad \ldots \quad q_k
\]

then it is known (see, e.g., [26]) that curves corresponding to ends of this splice diagram are cut out by polynomials (namely certain “approximate roots” g_i(x, y) of g(x, y)). It is easy to check that the functions g_i(x, y) then cut out curves in the surface zn = g(x, y) corresponding to the ends of its splice diagram

\[
p_1 \quad n \quad p_2 \quad n \ldots \quad p_k \quad n \quad q_1 \\
\quad q_1 \quad q_2 \quad \ldots \quad q_k
\]

so Theorem 8.1 applies. □

9. Appendix: Splicing and plumbing

In this appendix we recall the classification of \(\mathbb{Z}\)-homology sphere singularity links in terms of splice diagrams and describe how to recover a resolution diagram from the splice diagram.

We start with Seifert fibered manifolds. For the following results see [20]. Let \(\Sigma\) be a Seifert fibered homology 3-sphere other than \(S^3\). Then it has at least 3 singular fibers and the degrees \(p_1, \ldots, p_r\) of these singular fibers are pairwise coprime. Conversely, given a set \(\{p_1, \ldots, p_r\}\) of pairwise coprime integers \(p_i > 1\) with \(r \geq 3\), there is a unique Seifert fibered homology sphere \(\Sigma(p_1, \ldots, p_r)\) up to orientation with these singular fiber degrees. Moreover, \(\Sigma(p_1, \ldots, p_r)\) has a unique
orientation for which it is a singularity link, so we give it this orientation. It is, in
fact, the link of the Brieskorn-Pham complete intersection
\[ V(p_1, \ldots, p_r) := \{(z_1, \ldots, z_r) \in \mathbb{C}^n : a_{i1}z_1^{p_1} + \cdots + a_{ir}z_r^{p_r} = 0 \text{ for } i = 1, \ldots, r - 2 \}, \]
for a sufficiently general matrix \((a_{ij})\) of coefficients. By Hamm [9], “sufficiently
general” means that all \((r - 2) \times (r - 2)\) minors should be non-singular.

We represent the homology sphere \(\Sigma(p_1, \ldots, p_r)\) by the splice diagram

\[
\begin{array}{c}
\bullet & \cdots & \cdots & \bullet \\
& & & \\
& & & \\
p_r & p_{r-1} & \cdots & p_2 & p_1
\end{array}
\]

Each of the singular fibers of \(\Sigma(p_1, \ldots, p_r)\) represents a knot in \(\Sigma(p_1, \ldots, p_r)\)
which we represent in a splice diagram by adding an arrowhead to the corresponding
edge. Thus

\[
\begin{array}{c}
\bullet & \cdots & \cdots & \bullet \\
& & & \\
& & & \\
5 & 3 & 2
\end{array}
\]

represents the link in \(\Sigma(2, 3, 5)\) consisting of the degree 2 and 3 singular fibers. Non-
singular fibers are represented by adding new arrows at the central vertex weighted
by 1, so

\[
\begin{array}{c}
\bullet & \cdots & \cdots & \bullet \\
& & & \\
& & & \\
5 & 3 & 2
\end{array}
\]

represents the knot in \(\Sigma(2, 3, 5)\) consisting of one non-singular fiber.

There are Seifert fibrations of the 3-sphere with 2 or less singular fibers. For
instance, \(S^3\) can be fibered by copies of the \((p, q)\) torus knot, with one \(p\)-fold
singular fiber and one \(q\)-fold singular fiber, so the splice diagram

\[
\begin{array}{c}
\bullet & \cdots & \cdots & \bullet \\
& & & \\
& & & \\
1 & q & \cdots & p
\end{array}
\]

is the diagram for the \((p, q)\) torus knot in \(S^3\). Similarly

\[
\begin{array}{c}
\bullet & \cdots & \cdots & \bullet \\
& & & \\
& & & \\
q & 1 & \cdots & 1
\end{array}
\]

represents a pair of parallel \((1, q)\) torus knots (unknotted curves which link each
other \(q\) times).

If \(K_1 \subset \Sigma_1\) is a knot in a homology sphere and \(K_2 \subset \Sigma_2\) is another, then we
form the splice of \(\Sigma_1\) to \(\Sigma_2\) along \(K_1\) and \(K_2\) as follows. Let \(N_i\) be a closed tubular
neighborhood of \(K_i\) in \(\Sigma_i\) for \(i = 1, 2\) and let \(\Sigma'_i\) be the result of removing its
interior, so \(\partial \Sigma'_i = T^2\). The splice is the manifold

\[
\Sigma = \Sigma'_1 \cup_{T^2} \Sigma'_2,
\]

where the glueing matches meridian in \(\Sigma_1\) to longitude in \(\Sigma_2\) and vice versa. (“Merid-
ian” and “longitude” in \(\Sigma_1\) are the simple curves in \(\partial \Sigma'_1 = T^2\) that are null-
omologous respectively in the removed solid torus \(N_1\) or in \(\Sigma'_1\).) We denote the
splice by

\[ \Sigma = \Sigma_1 K_1 K_2 \Sigma_2. \]

We represent splicing in terms of splice diagrams by gluing the diagrams at the arrowheads that represent the knots along which we are splicing. For instance,

![splice diagram]

represents the splice of two copies of \( \Sigma(2, 3, 7) \) along the knots represented by the degree 7 fibers.

As described in Section 1, the splice diagrams that classify homology sphere singularity links are precisely the splice diagrams with pairwise coprime positive weights around each node and with positive edge determinants (recall that the edge determinant is the product of the two weights on the edge minus the product of the weights adjacent to the edge).

The splice diagram for a homology sphere singularity link can be computed very easily from a resolution diagram for the singularity. Let \((X, o)\) be a normal surface singularity germ and \(\Sigma\) its link, that is, the boundary of a regular neighborhood of \(o\) in \(X\). Assume for the moment only that \(\Sigma\) is a rational homology sphere, that is \(H_1(\Sigma)\) is finite.

Let \(\pi: \overline{X} \to X\) be a good resolution. “Good” means that the exceptional divisor \(E = \pi^{-1}(o)\) has only normal crossings. The rational homology sphere condition implies that \(E\) is rationally contractible; that is,

- each component of \(E\) is a smooth rational curve;
- the dual resolution graph \(T\) (the graph with a vertex for each component of \(E\) and an edge for each intersection of two components) is a tree.

We weight each vertex \(v\) of \(T\) by the self-intersection number \(E_v \cdot E_v\) of the corresponding component \(E_v\) of \(E\). The intersection matrix for \(T\) is the matrix \(A(T)\) with entries \(a_{vw} = E_v \cdot E_w\), that is,

- \(a_{vw} = 1\) if \(v \neq w\) and \(v\) and \(w\) are joined by an edge
- \(a_{vw} = 0\) if \(v \neq w\) and \(v\) and \(w\) are not joined by an edge
- \(a_{vv} = E_v \cdot E_v\)

It is well known that \(A(T)\) is negative-definite and its cokernel (also called the discriminant group) is \(H_1(\Sigma)\). In particular,

\[ d(T) := \det(-A(T)) \]

is the order of \(H_1(\Sigma)\).

We now assume that \(\Sigma\) is an \(\mathbb{Z}\)-homology sphere, so \(d(T) = 1\), that is, \(A(T)\) is unimodular. The splice diagram \(\Delta\) for \(\Sigma\) has the same overall shape as the resolution graph \(T\); it’s underlying graph is obtained from \(T\) by suppressing valency two vertices. The weights on edges are computed by the following procedure: At a vertex \(v\) of \(\Delta\) let \(T_{ev}\) be the subgraph of \(T\) cut off by the edge of \(T\) at \(v\) in the direction of \(e\), as in the following picture. The corresponding weight is then
Example 6. Here is an example of a resolution graph with homology sphere link. The reader can check that \( A(T) \) is negative definite and unimodular.

\[
T = \begin{pmatrix}
-2 & -1 & -17 & -1 \\
-3 & & & -3 \\
& & & -2
\end{pmatrix}
\]

Its splice diagram is:

\[
\Delta = \begin{pmatrix}
2 & 7 & 11 & 2 \\
3 & & & 5 \\
& & & 11
\end{pmatrix}
\]

For example, the weight 7 on the left node of \( \Delta \) is \( d(T_{ve}) \) with

\[
T_{ve} = \begin{pmatrix}
-17 & -1 \\
-3 & & -3 & -2
\end{pmatrix}
\]

An algorithm to recover the resolution diagram from the splice diagram is given in [7]. Here we describe an easier method that arose from conversations with Paul Norbury (developed independently by Pierrette Cassou-Nogues [3], whose terminology of “maximal splice diagram” we have adopted—we called it “adjoint diagram”). The maximal splice diagram is simply the version of the splice diagram that we get from the resolution graph if we do not first eliminate vertices of valency 2, and include edge weights at all vertices — also the leaves. Thus, for the above example, the maximal splice diagram is:

\[
\Delta' = \begin{pmatrix}
11 & 2 & 7 & 1 & 11 & 28 \\
5 & 3 & & & 5 & 9
\end{pmatrix}
\]

To compute the resolution graph from the splice diagram we will give algorithms to:

- compute the resolution graph from the maximal splice diagram, and
- compute the maximal splice diagram from the splice diagram.

We will need the following properties of the maximal splice diagram, which we prove later.

**Theorem 9.1.** 1. For any pair of vertices \( v \) and \( w \) of the maximal diagram let \( \ell_{vw} \) be the product of the weights adjacent to, but not on, the shortest path from \( v \) to \( w \) in \( \Delta' \). Then the matrix \( L := (\ell_{vw}) \) is the inverse matrix of \( -A(T) \).

2. Every edge determinant for the maximal splice diagram is 1.

3. The edge-weight adjacent to a leaf \( v \) of the maximal splice diagram is equal to \( \lceil a/b \rceil \) where \( a \) is the product of edge-weights adjacent to and just beyond the nearest node to \( v \) and \( b \) is the remaining weight adjacent to that node.
We remark that part 3. is valid also for the valency 2 vertices between the leaf and its nearest node. For example, for the right-most leaf of the above example $5 = \lceil \frac{22}{5} \rceil = \lceil \frac{9}{2} \rceil$.

9.1. **Resolution graph from maximal splice diagram.** The only issue is to recover the self-intersection weights $e_v := a_{vv}$ at vertices. The matrix equation $LA(T) = -I$ gives equations that will do this. We use the notation $w\rightarrow v$ to mean vertices $w$ and $v$ are connected by an edge. Then for any vertex $w'$ adjacent to $v$, the $vw'$ entry of this matrix equation gives:

$$e_v = \frac{-1}{\ell_{vw'}} \left( \sum_{\{w:w\rightarrow v\}} \ell_{ww'} \right).$$

Note that the product of the weights just beyond $w'$ from $v$ cancel in this formula, so they may be replaced by 1 for the calculation. For example, for

```
2 <- 7 <- 3,2 <- 5,3 <- 7,4 <- 9,5 <- 11 <- 2,3
```

we get the resolution graph string

```
-5 <- -2 <- -2 <- -2 <- 7,3
```

9.2. **Maximal splice diagram from splice diagram.** We describe how to recover the string of vertices and weights of the maximal splice diagram between any two vertices of a splice diagram. Suppose first both vertices are nodes with weights as follows,

```
\[ \begin{array}{ccccccccc}
a_1 & \cdots & a_r & b & c & \cdots & d_1 & \cdots & d_s \\
\end{array} \]
```

and put $a = \prod_{i=1}^r a_i$, $d = \prod_{j=1}^s d_j$. If one of the vertices (say the right one) is a leaf instead of a node then we put $d = 1$. The desired string of vertices and weights between our two nodes will only depend on $a, b, c, d$, so we replace the above diagram by

```
\[ \begin{array}{ccccccc}
a \wedge b & \cdots & c \wedge d \\
\end{array} \]
```

Consider the following infinite linear graph:

```
\[ \begin{array}{cccccccccc}
\cdots & 1 & 3 & \cdots & 1 & 2 & \cdots & 1 & 1 & \cdots & 3 & 1 & \cdots \\
\end{array} \]
```

We are going to refine this by adding vertices on this line until our vertices $a \wedge b$ and $c \wedge d$ appear on it. Vertices $x \wedge y$ are ordered along the line by size of $x/y$. Thus such a vertex either is already a vertex of the linear graph, or it falls on an existing edge. In the latter case we subdivide the edge as follows:

```
\[ \begin{array}{cccccccc}
a \beta & \gamma & \delta \\
\end{array} \]
```

We repeat this process until both our desired vertices appear, and then the portion of the linear graph between them is what we were seeking.
For example, here is the process to create the string for
\[ 10 \cdot 7 \cdot 11 \cdot 6 \cdot \]

We mark the positions of these vertices, until they are found, by \( \lor \).

Thus the final string is:
\[ 10 \cdot 7 \cdot 3 \cdot 2 \cdot 5 \cdot 3 \cdot 7 \cdot 4 \cdot 9 \cdot 5 \cdot 11 \cdot 6 \cdot \]

which, by the previous subsection, gives the resolution graph string
\[ -5 \cdot -2 \cdot -2 \cdot -2 \cdot \]

9.3. Proofs. We give proofs of Theorem 9.1 and of the above procedures. Property 1 of the theorem is easily shown by computing the adjoint matrix of \(-A(T)\); it is carried out in Lemma 20.2 of [7].

For property 2 of Theorem 9.1, suppose we have an edge connecting vertices \( v \) and \( w \) of the maximal splice diagram as follows,
The fact that the string of the maximal splice diagram between two vertices of a splice diagram only depends on $a, b, c, d$ is immediate from the discussion in [7]. Consider the resolution graph
\begin{equation}
\begin{array}{ccccccccccc}
\end{array}
\end{equation}
with associated maximal splice diagram
\begin{equation}
\begin{array}{ccccccccccc}
s & 1 & a & -1 & s & 1 & s & 1 & 2 & 1 & 3 & 1 & 4 & 1 & s & t & \cdots
\end{array}
\end{equation}
This is a piece of the infinite linear graph we used above, and we choose $s$ and $t$ large enough that our desired vertices will lie in this piece. Now we repeatedly blow up on edges of the linear resolution graph. An easy calculation shows that blowing up on an edge:
\begin{equation}
\begin{array}{c}
e_1 \\
-1
\end{array}
\begin{array}{c}
e_2
\end{array}
\begin{array}{c}
e_1
\begin{array}{c}
-1
\end{array}
\begin{array}{c}
-1
\end{array}
\end{equation}
has the effect
\begin{equation}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\beta
\end{array}
\begin{array}{c}
\gamma
\end{array}
\begin{array}{c}
\delta
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\beta
\end{array}
\begin{array}{c}
\gamma
\end{array}
\begin{array}{c}
\delta
\end{array}
\end{equation}
on the associated maximal splice diagram. Thus we need only show that our desired vertices eventually appear in this procedure. But this is a standard fact about Farey sequences (alternatively, one can observe that we are describing the standard procedure to resolve the plane curve singularity $(x^a + y^b)(x^c + y^d)$).

This same argument applies to see how to fill in the maximal splice diagram between a node and a leaf, even if the edge weight at the leaf is unknown. The leaf will be the rightmost vertex of the above string (12) with $t$ chosen as small as possible to accommodate our desired vertex $a \wedge b$. Thus, the $t$ that we choose is $[a/b]$ (if $t = 1$ the initial resolution string (11) is
\begin{equation}
\begin{array}{c}
-1 \\
-2
\end{array}
\begin{array}{c}
-2
\end{array}
\begin{array}{c}
-2
\end{array}
\end{equation}

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