HONORS MATH B: 1st PROBLEM SET

due Tuesday, January 28, 2014 in class

1. For each of the following sequence, find an $N_0 \in \mathbb{N}$ such that $|x_n| < \frac{1}{100}$ whenever $n \geq N_0$.
   
   (a) $x_n = \frac{1}{n^2 + 1}$;
   
   (b) $x_n = \sqrt{1 + n^2} - n$;
   
   (c) $x_n = \frac{5^n}{n!}$.

2. Which of the following subsets $A$ of $\mathbb{Q}$ are bounded above? If $A$ is bounded above, what is the least upper bound (as a real number)? Is the least upper bound rational? Is it an element of $A$?
   
   (a) $A = \{ x \in \mathbb{Q} : |x - 5| < 3 \}$;
   
   (b) $A = \{ \frac{1}{n} : n \in \mathbb{N} \}$;
   
   (c) $A = \{ n - \frac{1}{n} : n \in \mathbb{N} \}$;
   
   (d) $A = \{ x \in \mathbb{Q} : x^3 < 7 \}$.

3. Let $A \subset \mathbb{R}$. Prove that $C \in \mathbb{R}$ is the least upper bound of $A$ if and only if $C$ is an upper bound and for every $\epsilon > 0$, there is an $x \in A$ such that $x > C - \epsilon$.

4. Let $A \subset \mathbb{R}$ and $C = \text{sup} A$. Show that there is a sequence $\{x_n\}$ such that each $x_n \in A$ and $\{x_n\}$ converges to $C$.

5. Let $A \subset B \subset \mathbb{R}$ and $B$ is bounded above, show that $\text{sup} A \leq \text{sup} B$.

6. Let $A$ and $B$ be subsets of $\mathbb{R}$ and denote $A+B = \{x+y : x \in A \text{ and } y \in B \}$. Show that $\text{sup}(A+B) = \text{sup} A + \text{sup} B$. 