1. (20 points, this is exercise 4.12 in Steinberg). Find all irreducible representation of the quaternionic group \( Q_8 \) and write down the character table of \( Q_8 \).

(a) Start by recalling generators \( i, j, k \) of \( Q_8 \) and defining relations. Check that the map \( \rho \) in (Steinberg, exercise 4.12) gives a representation of \( Q_8 \) and compute its character \( \chi_\rho \). Check irreducibility of this representation by verifying \( (\chi_\rho, \chi_\rho) = 1 \).

(b) Find the commutator subgroup \([Q_8, Q_8]\) and the quotient \( Q_8/[Q_8, Q_8] \) by the commutator. Classify one-dimensional representations of \( Q_8 \).

(c) Determine conjugacy classes in \( Q_8 \).

(d) Write down the character table of \( Q_8 \) and the unitary matrix associated to the character table. Check orthogonality relations of the 2nd kind for several pairs of columns (this is a quick way to scan for possible errors).

(e) Compare this character table with the character table for the dihedral group \( D_8 \) obtained in class.

(Optional:) Find a natural construction of the homomorphism \( \rho \) from the realization of \( Q_8 \) as a subgroup of \( \mathbb{H}^* \), where \( \mathbb{H} \) is the ring of quaternions.

2. (10 points) (a) Representation \( W \) of the symmetric group \( S_3 \) has character

\[
\chi_W(1) = 7, \quad \chi_W((12)) = -1, \quad \chi_W((123)) = 4.
\]

Find multiplicities of irreducible representations of \( S_3 \) in \( W \) and its dimension.

(b) Does there exist a representation of \( S_3 \) with the character

\[
\chi_V(1) = 2, \quad \chi_V((12)) = 2, \quad \chi_V((123)) = -1?
\]

3.(10 points) (a) Character \( \chi_V \) of a complex representation of a finite group \( G \) has inner product \( (\chi_V, \chi_V) = 2 \). What can you say about decomposition of \( V \) into irreducibles? Same question if \( (\chi_V, \chi_V) = 3 \).

(b) Suppose that (complex) representation \( V \) of a finite group \( G \) has \( (\chi_V, \chi_W) = 0 \) for any irreducible representation \( W \) of \( G \). What can you say about \( V \)?