Algebraic topology

Discussion 5 Section 4.2 of Hatcher (subsections on the Freudenthal theorem, Eilenberg-MacLane spaces, fiber bundles).

Homework 5. Due Monday, October 16.

1. Show that the spaces $\mathbb{S}^2$ and $\mathbb{S}^3 \times \mathbb{C}P^\infty$ have isomorphic homotopy groups but are not homotopy equivalent.

2. Let $X_n$ be the subset of $\mathbb{C}^n$ consisting of $n$-tuples of pairwise distinct complex numbers $(z_1, \ldots, z_n)$.
   
   (a) Show that the forgetful map $X_n \to X_{n-1}$ given by omitting $z_n$ make $X_n$ a fiber bundle with base $X_{n-1}$.

   (b) Using induction on $n$ and the long exact homotopy sequence of a fibration prove that $X_n$ is a $K(H, 1)$-space for some group $H$. This group is called the pure braid group on $n$ strands.

   (c) The symmetric group $S_n$ acts on $X_n$ by permuting the $z_i$’s. Let $X_n \to X'_n$ be the quotient map. Explain why this map is a locally-trivial covering, compute its group of deck transformations, and show that $X'_n$ is a $K(G, 1)$-space for some group $G$ (called the braid group on $n$ strands).

   (d) Show that there exists a short exact sequence
   
   $$1 \to H \to G \to S_n \to 1.$$  

   (e) Determine the braid group and the pure braid group in the case $n = 2$.

3. Let $p : E \to B$ be a Serre fibration, $b \in B$ and $F = p^{-1}(b)$. We constructed a homomorphism $p_* : \pi_n(E, F) \to \pi(B)$ and checked its injectivity. Check that the homomorphism is surjective and conclude that $p_*$ is an isomorphism.

4. Suppose that $p : E \to B$ is a locally-trivial covering. What does the long exact sequence for $p$ tell us about the homotopy groups of $E, B,$ and $F$?

5. Let $G$ be a group and $X$ a simply-connected space. Show that for the product $K(G, 1) \times X$ the action of $\pi_1$ on $\pi_n$ is trivial for all $n > 1$. 