Solutions to Extra Problems

1. We consider the difference quotient
\[ \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} = \lim_{\Delta z \to 0} nz^{n-1} + \lim_{\Delta z \to 0} \Delta z \cdot Q(z, \Delta z) \]
where \( Q(z, \Delta z) \) is a polynomial of degree \( n - 2 \) in \( z \) and \( \Delta z \). Clearly, the limit equals \( nz^{n-1} \).

2. We have
\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \]
Differentiating formally, i.e., term-by-term yields
\[ (e^z)' = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z. \]

3. In all three cases we apply the Cauchy residue formula:
\[ \int_{C} \frac{\cos(z)}{z} \, dz = 2\pi i \cos(0) = 2\pi i. \]
\[ \int_{\partial C + \mathbb{C}} \frac{z^2 + 3z + 7}{z - 2} \, dz = 2\pi i (2^2 + 3 \cdot 2 + 7) = 34\pi i. \]
\[ \int_{C} \frac{z^3 + 1}{z} \, dz = 2\pi i (0^3 + 1) = 2\pi i. \]

4. \( f(z) = u(x, y) + iv(x, y) \) with \( u \) and \( v \) real-valued. Since \( f \) is holomorphic \( u, v \) satisfy the Cauchy-Riemann equations:
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \]
Thus,
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left( -\frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right). \]
Using the equality of the cross partials of \( v \), we see that the above expression vanishes. The other equation is established in a similar manner.

5. We have
\[ \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = F'(z). \]
Restricting to \( \Delta z \) real, i.e., \( \Delta z = \Delta x + i\Delta y \) with \( \Delta y = 0 \), and expanding out the real and imaginary parts we see,
\[ \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} = F'(z). \]
Clearly, from the definition of the partial derivatives for \( u \) and \( v \), this gives
\[ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = F'. \]
The second equation follows from this by applying the Cauchy-Riemann equations, or directly by computing the limits along the line $\Delta x = 0$.

6. Let us directly evaluate the integral for any $r > 0$. Parameterizing the circle as $re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we have

$$\int_{C_r} dz/z^2 = \int_0^{2\pi} i r e^{i\theta} d\theta \over (r^2 e^{2i\theta}) = \int_0^{2\pi} i d\theta \over r e^{i\theta} = -1 \over r e^{-i\theta} \bigg|_{0}^{2\pi} = 0.$$  

7. With the same parameterization as above we have:

$$\int_{C_r} dz/z^n = \int_0^{2\pi} i r e^{i\theta} d\theta \over (r^n e^{ni\theta}) = \int_0^{2\pi} i d\theta \over r^{n-1} e^{(n-1)i\theta} = \frac{1}{r^{n-1}} \int_0^{2\pi} i e^{-(n-1)i\theta} d\theta = -1 \over (n-1)r^{n-1} e^{-(n-1)i\theta} \bigg|_{0}^{2\pi} = 0.$$  

(Provided $n$ is an integer different from $-1$.)

8. We expand out:

\[
z^3 + 3z^2 + 2z + 7 = \frac{1}{z} + \frac{3}{z^2} + \frac{2}{z^3} + \frac{7}{z^4}.
\]

By the previous two problems the integrals of all the terms except the first around the unit circle vanish. By the Cauchy integral formula the first gives $2\pi i$. Thus,

$$\int_C \frac{z^3 + 3z^2 + 2z + 7}{z^4} dz = 2\pi i.$$  

9. We have $F(z) = U(x,y) + iV(x,y)$ has derivative $F'(z) = f(z)$ with $f(z) = u(x,y) + iv(x,y)$. We also have:

$$\int_\gamma f(z)dz = \int_\gamma (udx - vdy) + i(vdx + udy).$$  

By Problem 5, we have

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = u$$

and

$$\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} = v.$$  

Thus, $\nabla U = (u, -v)$ and $\nabla V = (v, u)$. Applying the fundamental theorem for line integrals to the real and imaginary parts we see that

$$\int_\gamma (udx - vdy) + i(vdx + udy) = U(\gamma(1)) - U(\gamma(0)) + i(V(\gamma(1)) - V(\gamma(0))) = (U + iV)(\gamma(1)) - (U + iV)(\gamma(0)) = F(\gamma(1)) - F(\gamma(0)).$$