Review Solutions
MAT V1102

1. (a) If \( u = 4 - x \), then \( du = -dx \). Hence, substitution implies

\[
\int \frac{1}{\sqrt{4 - x}} \, dx = - \int \frac{1}{\sqrt{u}} \, du = -2\sqrt{u} + C = -2\sqrt{4 - x} + C.
\]

(b) If \( u = e^t + e^{-t} \), then \( du = (e^t - e^{-t})\,dt \). Thus, by substitution, we have

\[
\int \frac{e^t - e^{-t}}{e^t + e^{-t}} \, dt = \int \frac{1}{u} \, du = \ln |u| + C = \ln |e^t + e^{-t}| + C.
\]

(c) If \( u = \sin(z^2) \), then \( du = 2z \cos(z^2) \, dz \). Therefore, substitution gives

\[
\int z \cos(z^2) \sqrt{\sin(z^2)} \, dz = \int \frac{1}{2\sqrt{u}} \, du = \sqrt{u} + K = \sqrt{\sin(z^2)} + K.
\]

(d) If \( u = (\ln w)^2 \) and \( dv = dw \), then \( du = \frac{2\ln w}{w} \, dw \) and \( v = w \). Hence, integration by parts yields

\[
\int (\ln w)^2 \, dw = (\ln w)^2 w - \int w \cdot \frac{2\ln w}{w} \, dw = (\ln w)^2 w - 2 \int \ln w \, dw.
\]

To compute \( \int \ln w \, dw \), we use integration by parts a second time. If \( \tilde{u} = \ln w \) and \( d\tilde{v} = dw \), then \( d\tilde{u} = \frac{1}{w} \, dw \) and \( \tilde{v} = w \). Thus

\[
\int (\ln w) \, dw = (\ln w)w - \int w \cdot \frac{1}{w} \, dw = (\ln w)w - w + C.
\]

Combining this with our first result gives

\[
\int (\ln w)^2 \, dw = (\ln w)^2 w - 2w(\ln w) + 2w + C.
\]

(e) If \( u = \sqrt{2 + 3y} \), then \( u^2 = 2 + 3y \) and \( 2u \, du = 3 \, dy \). It follows that \( dy = \frac{2u}{3} \, du \) \n\text{and} \ y = \frac{u^2 - 2}{3} \) which implies that \( y + 2 = \frac{u^2 - 2}{3} + 2 = \frac{u^2 + 4}{3} \). By making a substitution, we have

\[
\int (y + 2)\sqrt{2 + 3y} \, dy = \int \frac{u^2 + 4}{3} \cdot u \cdot \frac{2u}{3} \, du = \int \frac{2u^4 + 8u^2}{9} \, du = \frac{2u^5}{45} + \frac{8u^3}{27} + C = \frac{2(2 + 3y)^{5/2}}{45} + \frac{8(2 + 3y)^{3/2}}{27} + C.
\]

(f) Observe that \( x^2 + 6x + 25 = (x^2 + 6x + 9) + 16 = (x + 3)^2 + 16 \). If \( x + 3 = 4u \), then \( dx = 4du \) and \( (x + 3)^2 + 16 = 16(u^2 + 1) \). By substitution, we have

\[
\int \frac{1}{x^2 + 6x + 25} \, dx = \int \frac{1}{16(u^2 + 1)} \cdot 4du = \frac{\arctan(u)}{4} + C = \frac{1}{4} \arctan \left( \frac{x + 3}{4} \right) + C.
\]
(g) If \( u = w^2 \) and \( dv = \sin(3w) \, dw \), then \( du = 2w \, dw \) and \( v = -\frac{\cos(3w)}{3} \). Using integration by parts, we have

\[
\int w^2 \sin(3w) \, dw = -w^2 \cdot \frac{\cos(3w)}{3} + \int 2w \cdot \frac{\cos(3w)}{3} \, dw = -\frac{w^2 \cos(3w)}{3} + \frac{2}{3} \int w \cos(3w) \, dw .
\]

To compute \( \int w \cos(3w) \, dw \), we use integration by parts once again. If \( \tilde{u} = w \) and \( d\tilde{v} = \cos(3w) \, dw \), then \( d\tilde{u} = dw \) and \( \tilde{v} = \frac{\sin(3w)}{3} \), whence

\[
\int w \cos(3w) \, dw = w \cdot \frac{\sin(3w)}{3} - \int \frac{\sin(3w)}{3} \, dw = \frac{w \sin(3w)}{3} + \frac{\cos(3w)}{9} \, dw .
\]

Therefore, we have

\[
\int w^2 \sin(3w) \, dw = -\frac{w^2 \cos(3w)}{3} + \frac{2}{3} \int w \sin(3w) + \frac{\cos(3w)}{9} \, dw .
\]

(h) If \( u = 2z - z^2 \), then \( du = (2 - 2z) \, dz \) and substitution gives

\[
\int \frac{z - 1}{\sqrt{2z - z^2}} \, dz = -\int \frac{1}{2\sqrt{u}} \, du = -\sqrt{u} + C = -\sqrt{2z - z^2} + K .
\]

(i) If \( u = \ln t \) and \( dv = t^3 \, dt \), then \( du = \frac{1}{t} \, dt \) and \( v = \frac{t^4}{4} \). Integration by parts implies

\[
\int t^3 \ln t \, dt = (\ln t) \cdot \frac{t^4}{4} - \int \frac{t^4}{4} \cdot \frac{1}{t} \, dt = \frac{t^4 \ln t}{4} - \frac{t^4}{16} + C .
\]

(j) First, we have

\[
\frac{y^4 + 3y^3 + 2y^2 + 1}{y^2 + 3y + 2} = \frac{y^2(3y + 2) + 1}{y^2 + 3y + 2} = y^2 + \frac{1}{y^2 + 3y + 2}
\]

To find a partial fraction decomposition, we write

\[
\frac{1}{y^2 + 3y + 2} = \frac{A}{y + 1} + \frac{B}{y + 2} = \frac{A(y + 2) + B(y + 1)}{(y + 1)(y + 2)}
\]

which implies that \( 1 = A(y + 2) + B(y + 1) = (A + B)y + (2A + B) \). Since the coefficients of the various polynomials must be equal, we obtain \( A + B = 0 \) and \( 2A + B = 1 \). Solving these two equations gives \( A = 1 \) and \( B = -1 \). Hence,

\[
\frac{y^4 + 3y^3 + 2y^2 + 1}{y^2 + 3y + 2} = y^2 + \frac{1}{y + 1} - \frac{1}{y + 2} .
\]
Thus
\[ \int \frac{y^4 + 3y^3 + 2y^2 + 1}{y^2 + 3y + 2} \, dy = \int \frac{y^2 + \frac{1}{y + 1} - \frac{1}{y + 2} \, dy}{y^2 + 3y + 2} \]
\[ = \frac{y^3}{3} + \ln |y + 1| - \ln |y + 2| + C \]
\[ = \frac{y^3}{3} + \ln \left| \frac{y + 1}{y + 2} \right| + C . \]

(k) If \( u = e^t \), then \( du = e^t \, dt \) and substitution gives
\[ \int \frac{e^t \, dt}{e^{2t} + 3e^t + 2} = \int \frac{du}{u^2 + 3u + 2} . \]
To find a partial fraction decomposition, we write
\[ \frac{1}{u^2 + 3u + 2} = \frac{A}{u + 1} + \frac{B}{u + 2} = \frac{A(u + 2) + B(u + 1)}{(u + 1)(u + 2)} \]
which implies that \( 1 = A(u + 2) + B(u + 1) \). Setting \( u = -1 \) implies \( A = 1 \) and setting \( u = -2 \) implies \( B = -1 \). Thus,
\[ \int \frac{du}{u^2 + 3u + 2} = \int \frac{1}{u + 1} - \frac{1}{u + 2} \, du \]
\[ = \ln |u + 1| - \ln |u + 2| + K = \ln \left( \frac{e^t + 1}{e^t + 2} \right) + K . \]

(l) If \( u = \cos \theta \), then \( du = -\sin \theta \, d\theta \) and substitution yields
\[ \int \frac{\sin \theta \, d\theta}{\cos^2 \theta + \cos \theta - 2} = \int \frac{-du}{u^2 + u - 2} . \]
To find a partial fraction decomposition, we write
\[ \frac{-1}{u^2 + u - 2} = \frac{A}{u + 2} + \frac{B}{u + 1} = \frac{A(u + 1) + B(u + 2)}{(u + 1)(u + 2)} \]
which implies that \( -1 = A(u + 1) + B(u + 2) \). Setting \( u = -2 \) implies \( A = 1/3 \) and setting \( u = 1 \) implies \( B = -1/3 \). Thus,
\[ \int \frac{du}{u^2 + u - 2} = \int \frac{1}{3} \cdot \frac{1}{u + 2} - \frac{1}{3} \cdot \frac{1}{u + 1} \, du = \frac{1}{3} \ln \left| \frac{u + 2}{u + 1} \right| + C = \frac{1}{3} \ln \left| \frac{\cos \theta + 2}{\cos \theta - 1} \right| + C . \]

(m) First, observe that
\[ \frac{2x^3 - 2x^2 + 1}{x^2 - x} = 2x + \frac{1}{x^2 - x} = 2x + \frac{A}{x - 1} + \frac{B}{x} \]
which implies \( 1 = Ax + B(x - 1) \). Setting \( x = 1 \) implies \( A = 1 \) and setting \( x = 0 \) implies \( B = -1 \). Therefore, we have
\[ \int \frac{2x^3 - 2x^2 + 1}{x^2 - x} \, dx = \int 2x + \frac{1}{x - 1} - \frac{1}{x} \, dx = x^2 + \ln \left| \frac{x - 1}{x} \right| + C . \]
To find a partial fraction decomposition, we write
\[
\frac{2s + 2}{(s^2 + 1)(s - 1)^2} = \frac{A}{s - 1} + \frac{B}{(s - 1)^2} + \frac{Cs + D}{s^2 + 1}
\]
\[
= \frac{A(s - 1)(s^2 + 1) + B(s^2 + 1) + (Cs + D)(s - 1)^2}{(s^2 + 1)(s - 1)^2}
\]
\[
= \frac{A(s^3 - s^2 + s - 1) + B(s^2 + 1) + (Cs + D)(s^2 - 2s + 1)}{(s^2 + 1)(s - 1)^2}
\]
\[
= \frac{(A + C)s^3 + (-A + B - 2C + D)s^2 + (A + C - 2D)s + (-A + B + D)}{(s^2 + 1)(s - 1)^2}
\]
Comparing coefficients, we have
\[
\begin{cases}
A + C = 0 \\
-A + B - 2C + D = 0 \\
A + C - 2D = 2 \\
-A + B + D = 2.
\end{cases}
\]
The equations \(A + C = 0\) and \(A + C - 2D = 2\) imply that \(D = -1\) and \(C = -A\). Combining this with the equation \(-A + B - 2C + D = 0\), we have \(A + B = 1\). Similarly, the equations \(-A + B + D = 2\) and \(D = -1\) give \(-A + B = 3\). From \(A + B = 1\) and \(-A + B = 3\), we obtain \(A = -1\) and \(B = 2\). Therefore, \(A = -1\), \(B = 2\), \(C = 1\), \(D = -1\) and the partial fraction decomposition is
\[
\frac{2s + 2}{(s^2 + 1)(s - 1)^2} = -\frac{1}{s - 1} + \frac{2}{(s - 1)^2} + \frac{s - 1}{s^2 + 1}
\]
\[
= -\frac{1}{s - 1} + \frac{2}{(s - 1)^2} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}.
\]
Therefore, we have
\[
\int \frac{2s + 2}{(s^2 + 1)(s - 1)^2} \, ds = \int -\frac{1}{s - 1} + \frac{2}{(s - 1)^2} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1} \, ds
\]
\[
= -\ln|s - 1| - \frac{2}{s - 1} + \frac{1}{2} \ln(s^2 + 1) - \arctan(s) + C.
\]
To find a partial fraction decomposition, we write
\[
\frac{3z^2 + z + 4}{z^3 + z} = \frac{3z^2 + z + 4}{z(z^2 + 1)} = \frac{A}{z} + \frac{Bz + C}{z^2 + 1} = \frac{A(z^2 + 1) + z(Bz + C)}{z(z^2 + 1)} = \frac{(A + B)z^2 + Cz + A}{z(z^2 + 1)}.
\]
Comparing coefficients, we have \(A + B = 3\), \(C = 1\) and \(A = 4\) which implies \(A = 4\), \(B = -1\) and \(C = 1\). Thus, we obtain
\[
\frac{3z^2 + z + 4}{z^3 + z} = \frac{4}{z} + \frac{-z + 1}{z^2 + 1} = \frac{4}{z} - \frac{z}{z^2 + 1} + \frac{1}{z^2 + 1}.
\]
and
\[ \int_{1}^{\sqrt{3}} \frac{3z^2 + z + 4}{z^3 + z} \, dz = \int_{1}^{\sqrt{3}} \frac{4}{z} - \frac{z}{z^2 + 1} + \frac{1}{z^2 + 1} \, dz \]
\[ = \left[ 4 \ln |z| - \frac{1}{2} \ln |z^2 + 1| + \arctan(z) \right]_{1}^{\sqrt{3}} \]
\[ = 4 \ln \sqrt{3} - \frac{1}{2} \ln 4 + \arctan(\sqrt{3}) - \left( -\frac{1}{2} \ln 2 + \arctan(1) \right) \]
\[ = 2 \ln 3 - \frac{1}{2} \ln 2 + \frac{\pi}{12} . \]

(P) If \( u = x - 1 \), then \( du = dx \) and \( x = u + 1 \). Substitution implies
\[ \int_{-1}^{0} \frac{x^3 \, dx}{x^2 - 2x + 1} = \int_{2}^{-1} \frac{(u + 1)^3 \, du}{u^2} \]
\[ = \int_{2}^{-1} \frac{u^3 + 3u^2 + 3u + 1}{u^2} \, du \]
\[ = \left[ \frac{u^2}{2} + 3u + 3 \ln |u| - \frac{1}{u} \right]^{-1}_{-2} = 2 - 3 \ln 2 . \]

(Q) We have
\[ \int \frac{1 - w}{\sqrt{1 - w^2}} \, dw = \int \frac{1}{\sqrt{1 - w^2}} \, dw - \int \frac{w}{\sqrt{1 - w^2}} \, dw = \arcsin(w) + \sqrt{1 - w^2} + K . \]

(R) If \( u = 2y \), then \( du = 2 \, dy \) and substitution gives
\[ \int_{0}^{1/2} \frac{2 - 8y}{1 + 4y^2} \, dy = \int_{0}^{1/2} \frac{1 - 2u \, du}{1 + u^2} = \int_{0}^{1} \frac{du}{1 + u^2} - \int_{0}^{1} \frac{2u \, du}{1 + u^2} \]
\[ = \left[ \arctan(u) - \ln |1 + u^2| \right]_{0}^{1} = \frac{\pi}{4} - \ln 2 . \]

(S) Observe that \(-t^2 + 4t - 3 = -(t^2 - 4t + 4) + 1 = 1 - (t - 2)^2\).
If \( u = t - 2 \), then \( du = dt \) and we have
\[ \int \frac{dt}{\sqrt{-t^2 + 4t - 3}} = \int \frac{du}{\sqrt{1 - u^2}} = \arcsin(u) + C = \arcsin(t - 2) + C . \]

(T) If \( u = \arctan(x) \) and \( dv = dx \), then \( du = \frac{1}{1 + x^2} \, dx \) and \( v = x \). Hence, integration by parts yields
\[ \int \arctan(x) \, dx = x \arctan(x) - \int \frac{x}{1 + x^2} \, dx = x \arctan(x) - \frac{1}{2} \ln(1 + x^2) + K . \]
(u) If \( u = 2w + 1 \), then \( du = 2 \, dw \) and substitution gives
\[
\int \frac{4 \, dw}{1 + (2w + 1)^2} = \int \frac{2 \, du}{1 + u^2} = 2 \arctan(u) + C = 2 \arctan(2w + 1) + C.
\]

(v) If \( u = \ln z \), then \( du = \frac{1}{z} \, dz \). Observe that \( \ln 1 = 0 \) and \( \ln(e^{\pi/3}) = \pi/3 \). Hence, substitution gives
\[
\int_{1}^{e^{\pi/3}} \frac{dz}{z \cos(\ln(z))} = \int_{0}^{\pi/3} \frac{du}{\cos(u)} = \int_{0}^{\pi/3} \sec(u) \, du
\]
\[
= \ln |\sec(u) + \tan(u)|_{0}^{\pi/3} = \ln(2 + \sqrt{3}).
\]

(w) If \( u = \sqrt{y} \), then \( du = \frac{1}{2\sqrt{y}} \, dy \) and \( 2du = \frac{1}{\sqrt{y}} \, dy \). Observe that \( 1 + y = 1 + u^2 \). Substitution yields
\[
\int_{6}^{12} \frac{dy}{\sqrt{y}} = \int_{12}^{1} \frac{du}{2\sqrt{u}} = 12 \arctan(u) + C = 12 \arctan(\sqrt{y}) + C.
\]

(y) Recall that \( (27)^{2\theta+1} = e^{(2\theta+1)\ln(27)} \). If \( u = (2\theta + 1) \ln(27) \), then \( du = 2 \ln(27) \, d\theta \) and \( d\theta = \frac{du}{2 \ln(27)} \). Hence, substitution yields
\[
\int (27)^{2\theta+1} \, d\theta = \int \frac{e^{u}}{2 \ln 27} \, du = \frac{e^{u}}{2 \ln 27} + C = \frac{(27)^{2\theta+1}}{2 \ln 27} + C.
\]

(z) If \( u = \cos(2x) \), then \( du = -2\sin(2x) \, dx \) and \( \sin(2x) \, dx = -\frac{du}{2} \). Using substitution, we obtain
\[
\int \sin(2x)e^{\cos(2x)} \, dx = -\int \frac{1}{2} e^{u} \, du = -\frac{1}{2} e^{u} + C = -\frac{1}{2} e^{\cos(2x)} + C.
\]

2. (a) If \( u = x^n \) and \( dv = e^x \, dx \), then \( du = nx^{n-1} \, dx \) and \( v = e^x \). Integration by parts implies
\[
\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.
\]

(b) If \( u = x^n \) and \( dv = \cos(ax) \, dx \), then \( du = nx^{n-1} \, dx \) and \( v = \frac{\sin(ax)}{a} \). Using integration by parts, we have
\[
\int x^n \cos(ax) \, dx = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, dx.
\]
(c) If \( u = \cos^{n-1}(x) \) and \( dv = \cos(x)\, dx \), then \( du = -(n-1)\cos^{n-2}(x)\sin(x)\, dx \) and \( v = \sin(x) \). Applying integration by parts, we obtain

\[
\int \cos^n(x) \, dx = \cos^{n-1}(x)\sin(x) - \int \sin(x) \left( -(n-1)\cos^{n-2}(x)\sin(x) \right) \, dx \\
= \cos^{n-1}(x)\sin(x) + (n-1) \int \sin^2(x)\cos^{n-2}(x) \, dx \\
= \cos^{n-1}(x)\sin(x) + (n-1) \int \left( 1 - \cos^2(x) \right)\cos^{n-2}(x) \, dx \\
= \cos^{n-1}(x)\sin(x) + (n-1) \int \cos^{n-2}(x) \, dx - (n-1) \int \cos^n(x) \, dx.
\]

Therefore,

\[
n \int \cos^n(x) \, dx = \cos^{n-1}(x)\sin(x) + (n-1) \int \cos^{n-2}(x) \, dx.
\]

which implies

\[
\int \cos^n(x) \, dx = \frac{1}{n} \cos^{n-1}(x)\sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x) \, dx.
\]

3. (a) If \( u = 5x + 2 \), then \( du = 5\, dx \) and substitution gives

\[
\int_1^\infty \frac{1}{5x+2} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{5u} \, du = \lim_{b \to \infty} \left[ \frac{\ln|u|}{5} \right]_1^b = \lim_{b \to \infty} \frac{\ln|b|}{5} - \frac{\ln|1|}{5} = \infty.
\]

Therefore, the integral \( \int_1^\infty \frac{1}{5x+2} \, dx \) diverges.

(b) If \( u = t \) and \( dv = e^{-t}\, dt \), then \( du = dt \) and \( v = -e^{-t} \). Using integration by parts, we obtain

\[
\int_0^\infty te^{-t} \, dt = \lim_{b \to \infty} \left[ -te^{-t} \right]_0^b + \int_0^b e^{-t} \, dt \\
= \lim_{b \to \infty} \left[ -te^{-t} - e^{-t} \right]_0^b \\
= \lim_{b \to \infty} -be^{-b} - e^{-b} + 1 = 1.
\]

Here, we have used the fact that

\[
\lim_{b \to \infty} be^{-b} = \lim_{b \to \infty} \frac{b}{e^b} = \lim_{b \to \infty} \frac{1}{e^b} = 0
\]

and \( \lim e^{-b} = 0 \). Thus, the integral \( \int_0^\infty te^{-t} \, dt \) converges.

(e) By definition, we have

\[
\int_{-\infty}^\infty \frac{dz}{z^2 + 25} = \int_{-\infty}^0 \frac{dz}{z^2 + 25} + \int_0^\infty \frac{dz}{z^2 + 25}.
\]
If \( z = 5u \), then \( dz = 5 \, du \) and substitution gives
\[
\int_{0}^{\infty} \frac{dz}{z^2 + 25} = \lim_{b \to \infty} \int_{0}^{b} \frac{du}{5(u^2 + 1)} = \lim_{b \to \infty} \left[ \frac{\arctan(u)}{5} \right]_{0}^{b} = \lim_{b \to \infty} \frac{\arctan(b)}{5} = \frac{\pi}{10}.
\]
Here, we have used the fact that \( \lim \arctan(b) = \frac{\pi}{2} \). Finally, the substitution \( w = -z \) implies that
\[
\int_{-\infty}^{0} \frac{dz}{z^2 + 25} = \int_{0}^{\infty} \frac{-dw}{w^2 + 25} = \int_{0}^{\infty} \frac{dw}{w^2 + 25}.
\]
Therefore, \( \int_{-\infty}^{\infty} \frac{dz}{z^2 + 25} = \frac{\pi}{5} \).

(d) If \( u = \ln w \), then \( du = \frac{1}{w} \, dw \) and substitution gives
\[
\int_{2}^{\infty} \frac{dw}{w \ln w} = \lim_{b \to \infty} \int_{2}^{b} \frac{du}{u \ln u} = \lim_{b \to \infty} \frac{\ln |u|}{\ln 2} = \lim_{b \to \infty} \ln b - \ln(\ln 2) = \infty.
\]
Hence, the integral \( \int_{2}^{\infty} \frac{dw}{w \ln w} \) diverges.

(e) If \( u = \ln w \), then \( du = \frac{1}{w} \, dw \) and substitution gives
\[
\int_{1}^{2} \frac{dw}{w \ln w} = \int_{0}^{\ln 2} \frac{du}{\ln u} = \lim_{b \to 0^+} \int_{0}^{\ln 2} \frac{du}{\ln u} = \lim_{b \to 0^+} [\ln |u|]_{b}^{\ln 2} = \lim_{b \to 0^+} \ln (\ln 2) - \ln b = \infty,
\]
Therefore, the integral \( \int_{1}^{2} \frac{dw}{w \ln w} \) diverges.

(f) If \( x = 2u \), then \( dx = 2 \, du \) and substitution gives
\[
\int_{0}^{2} \frac{1}{\sqrt{4-x^2}} \, dx = \lim_{b \to 1^-} \int_{0}^{b} \frac{1}{\sqrt{1-u^2}} \, du = \lim_{b \to 1^-} [\arcsin(u)]_{0}^{b} = \lim_{b \to 1^-} \arcsin(b) = \arcsin(1) = \frac{\pi}{2}.
\]
Thus, the integral \( \int_{0}^{2} \frac{1}{\sqrt{4-x^2}} \, dx \) converges to \( \frac{\pi}{2} \).

(g) If \( u = \sqrt{y} \), then \( du = \frac{1}{2\sqrt{y}} \, dy \) and substitution gives
\[
\int_{0}^{\pi} \frac{1}{\sqrt{y}} e^{-\sqrt{y}} \, dy = \lim_{b \to 0^+} \int_{0}^{\pi} 2e^{-u} \, du = \lim_{b \to 0^+} -2e^{-\sqrt{\pi}} + 2e^{-\sqrt{b}} = 2 - 2e^{-\sqrt{\pi}}.
\]
Therefore the integral \( \int_{0}^{\pi} \frac{1}{\sqrt{4-x^2}} \, dx \) converges to \( 2 - 2e^{-\sqrt{\pi}} \).

(h) Observe that \( \frac{1}{(4-\theta)^2} \) is not defined at \( \theta = 4 \). Thus,
\[
\int_{3}^{6} \frac{d\theta}{(4-\theta)^2} = \lim_{a \to 4^-} \int_{3}^{a} \frac{d\theta}{(4-\theta)^2} + \lim_{b \to 4^+} \int_{b}^{6} \frac{d\theta}{(4-\theta)^2}.
\]
If \( u = 4 - \theta \), then \( du = -d\theta \) and substitution gives
\[
\lim_{b \to 4^+} \int_{b}^{6} \frac{d\theta}{(4-\theta)^2} = \lim_{c \to 0^+} \int_{c}^{2} -\frac{du}{u} = \lim_{c \to 0^+} \left[ \frac{1}{u} \right]_{c}^{-2} = \lim_{c \to 0^+} -\frac{1}{2} + \frac{1}{c} = \infty.
\]
Thus, \( \int_{3}^{6} \frac{d\theta}{(4-\theta)^2} \) diverges.
4. (a) The area of an equilateral triangles with sides of length \( \ell \) is \( \frac{1}{2} \sqrt{3} \ell^2 \). Hence, the cross-sectional area is \( A(x) = \frac{1}{2} \sqrt{3}(2\sqrt{\sin x})^2 = 2\sqrt{3} \sin x \) and

\[
\text{Volume} = \int_0^{\pi} A(x) \, dx = \int_0^{\pi} 2\sqrt{3} \sin x \, dx = \left[-2\sqrt{3} \cos x\right]_0^{\pi} = 4\sqrt{3}.
\]

(b) The area of an square with sides of length \( \ell \) is \( \ell^2 \). Thus, the cross-sectional area is \( A(x) = (2\sqrt{\sin x})^2 = 4 \sin x \) and

\[
\text{Volume} = \int_0^{\pi} A(x) \, dx = \int_0^{\pi} 4 \sin x \, dx = \left[-4 \cos x\right]_0^{\pi} = 8.
\]

5. (a) Since the area of a circle of radius \( r \) is \( \pi r^2 \), the cross-sectional area is \( A(x) = \pi (x^2)^2 = \pi x^4 \) and

\[
\text{Volume} = \int_0^{2} A(x) \, dx = \int_0^{2} \pi x^4 \, dx = \left[\frac{x^5}{5}\right]_0^{2} = \frac{32}{5}.
\]

(b) Since the area of a circle of radius \( r \) is \( \pi r^2 \), the cross-sectional area is \( A(x) = \pi (\sqrt{9 - x^2})^2 = \pi (9 - x^2) \) and

\[
\text{Volume} = \int_{-3}^{3} A(x) \, dx = \int_{-3}^{3} \pi (9 - x^2) \, dx = \pi \left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 36\pi.
\]

(c) Since the area of a circle of radius \( r \) is \( \pi r^2 \), the cross-sectional area is \( A(x) = \pi (e^{-x})^2 = \pi e^{-2x} \) and

\[
\text{Volume} = \int_{0}^{1} A(x) \, dx = \int_{0}^{1} \pi e^{-2x} \, dx = -\pi \left[\frac{e^{-2x}}{2}\right]_0^{1} = \frac{\pi}{2}(1 - e^{-2}).
\]

6. (a) Since the function is increasing, left-hand sum will give an underestimate and right-hand sum will give an overestimate.

(b) Since the function is concave-down, trapezoid rule will give an underestimate and the midpoint rule will give an overestimate.

(c) Since the function is concave-up, trapezoid rule will give an overestimate and the midpoint rule will give an underestimate. Furthermore, because the function is decreasing, left-hand sums will also give an overestimate and right-hand sums will give an underestimate.

(d) Since the function is concave-up, trapezoid rule will give an overestimate and the midpoint rule will give an underestimate.
7. (a) Since \( y = \frac{1}{3} (x^2 + 2)^{3/2} \), we have \( \frac{dy}{dx} = x\sqrt{x^2 + 2} = \sqrt{x^4 + 2x^2} \). Hence,

\[
\text{arc-length} = \int_0^3 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^3 \sqrt{1 + x^4 + 2x^2} \, dx = \int_0^3 \sqrt{(1 + x^2)^2} \, dx \\
= \int_0^3 x^2 \, dx = \left[ x + \frac{x^3}{3} \right]_0^3 = 12.
\]

(b) Since \( x = \frac{1}{2} (e^y + e^{-y}) \), we have \( \frac{dx}{dy} = \frac{1}{2} (e^y - e^{-y}) \) and

\[
\text{arc-length} = \int_{-1}^1 \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \int_{-1}^1 \sqrt{1 + \frac{1}{4} (e^{2y} - 2 + e^{-2y})} \, dy \\
= \int_{-1}^1 \sqrt{\frac{1}{4} (e^{2y} + 2 + e^{-2y})} \, dy = \int_{-1}^1 \frac{1}{2} \sqrt{(e^y + e^{-y})^2} \, dy \\
= \frac{1}{2} \int_{-1}^1 e^y + e^{-y} \, dy = \frac{1}{2} [e^y - e^{-y}]_{-1}^1 = e - e^{-1}.
\]

(c) Since \( y = \frac{x^4}{4} + \frac{1}{8x^2} \), we have \( \frac{dy}{dx} = x^3 - \frac{1}{4x^3} \) and

\[
\text{arc-length} = \int_{-1}^2 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dy = \int_{-1}^2 \sqrt{1 + \left( x^3 - \frac{1}{4x^3} \right)^2} \, dx \\
= \int_{-1}^2 \sqrt{x^6 + \frac{1}{2} + \frac{1}{16x^6}} \, dx = \int_{-1}^2 \sqrt{x^3 + \frac{1}{4x^3}}^2 \, dx \\
= \int_{-1}^2 x^3 + \frac{1}{4x^3} \, dx = \left[ \frac{x^4}{4} - \frac{1}{8x^2} \right]_1^2 = 123/32.
\]

(d) Since \( y = \ln(1 - x^2) \), we have \( \frac{dy}{dx} = -\frac{2x}{1 - x^2} \) and

\[
\text{arc-length} = \int_0^{1/2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^{1/2} \sqrt{1 + \left( \frac{2x}{1 - x^2} \right)^2} \, dx \\
= \int_0^{1/2} \sqrt{1 + \frac{4x^2}{(1-x^2)^2}} \, dx = \int_0^{1/2} \frac{x^2 + 4x^2 + 1}{(1-x^2)^2} \, dx \\
= \int_0^{1/2} \sqrt{x^2 + 1} \, dx = \int_0^{1/2} \frac{x^2 + 1}{1-x^2} \, dx = \int_0^{1/2} \left( -1 + \frac{1}{1-x} + \frac{1}{1+x} \right) \, dx \\
= [-x - \ln|x-1| + \ln|x+1|]_0^{1/2} = \ln 3 - 1/2.
\]

Here, we have used the partial fraction decomposition

\[
\frac{x^2 + 1}{1-x^2} = -1 + \frac{1}{1-x} + \frac{1}{1+x}.
\]
8. (a) The probability that you dropped your glove within one kilometer of home is
\[
\int_0^1 2e^{-2x} \, dx = \left[-e^{-2x}\right]_0^1 = 1 - e^{-2}.
\]
(b) The probability that you dropped it with \(y\) kilometers equals to 0.95 whence
\[
0.95 = \int_0^y 2e^{-2x} \, dx = \left[-e^{-2x}\right]_0^y = 1 - e^{-2y}.
\]
Solving \(e^{-2y} = 0.05\) for \(y\), we obtain \(y = -\frac{1}{2} \ln(0.05) \approx 1.5\) km.

9. (a) If \(Q_n\) represents the quantity, in milligrams, of cephalexin in the blood right after the \(n\)-th tablet, then we have
\[
Q_1 = 250
\]
\[
Q_2 = 250(0.01) + 250 = 252.5
\]
\[
Q_3 = 250(0.01)^2 + 250(0.01) + 250 = 252.525.
\]
(b) In general, we have
\[
Q_n = 250(0.01)^{n-1} + 250(0.01)^{n-2} + \cdots + 250(0.01) + 250
\]
\[
= 250 \left[1 + (0.01) + \cdots + (0.01)^{n-1}\right]
\]
\[
= \frac{250(1 - (0.01)^n)}{1 - 0.01}.
\]
(c) To determine the quantity of cephalexin in the body in the long run, we take the limit as \(n\) tends to infinity:
\[
\lim_{n \to \infty} Q_n = \lim_{n \to \infty} Q_n \frac{250(1 - (0.01)^n)}{1 - 0.01} = \frac{250}{0.99} = 252.5252525\ldots.
\]
Therefore, the quantity of cephalexin in the body in the long run is 252.53 milligrams.

10. This is a slightly encrypted geometric series:
\[
1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \cdots = 1 + \frac{237}{10^4} + \frac{237}{10^5} + \frac{237}{10^6} + \cdots
\]
\[
= 1 + \sum_{n=0}^{\infty} \frac{237}{1000} \cdot \left(\frac{1}{1000}\right)^n
\]
\[
= 1 + \frac{237}{1000} \cdot \frac{1}{1 - \frac{1}{1000}}
\]
\[
= 1 + \frac{237}{1000} \cdot \frac{1000}{999} = 1 + \frac{237}{999}.
\]

11. (a) Since \((\sqrt{2})^{1-n} = \sqrt{2} \left(\frac{1}{\sqrt{2}}\right)^n\), this is a geometric series. Because \(\sqrt{2}^{-1} < 1\), it converges.
(b) Since
\[
\frac{1+2^n+3^n}{5^n} = \left(\frac{1}{5}\right)^n + \left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n,
\]
this is a sum of three convergent geometric series. Hence \(\sum \frac{1+2^n+3^n}{5^n}\) converges.

(c) If \(f(x) = x e^{-x^2}\) then \(f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = (1-2x^2)e^{-x^2}\). When \(x > 1\), \(f'(x) < 0\) which implies \(f(x)\) is decreasing. Because \(f(x) = x e^{-x^2}\) is positive continuous and decreasing on \([1, \infty)\), we may apply the integral test:

\[
\int_1^\infty x e^{-x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{e^{-x^2}}{2} \right]_1^b = \lim_{b \to \infty} -b e^{-b^2} + e^{-1} = e^{-1}.
\]

Therefore, the series \(\sum n e^{-n^2}\) converges.

(d) If \(f(x) = \frac{1}{x \sqrt{\ln x}}\) then \(f'(x) = -\frac{1}{2} \left(\frac{2\ln(x)+1}{x^2(\ln(x))^{3/2}}\right)\). When \(x > 1\), \(f'(x) < 0\) which implies \(f(x)\) is decreasing. Because \(f(x) = \frac{1}{x \sqrt{\ln x}}\) is positive, continuous and decreasing on \([2, \infty)\), we may apply the integral test:

\[
\int_2^\infty \frac{1}{x \sqrt{\ln x}} \, dx = \lim_{b \to \infty} \left[ 2(\ln x)^{1/2} \right]_2^b = -2(\ln 2)^{1/2} + \lim_{b \to \infty} 2(\ln b)^{1/2} = \infty.
\]

Hence, the series \(\sum \frac{1}{n \sqrt{\ln n}}\) diverges.

(e) If \(f(x) = \frac{1}{x(\ln x)|\ln(\ln x)|^2}\) then \(f'(x) = -\frac{\ln x \cdot \ln(\ln x) + \ln(\ln x) + 2}{x^2 |\ln(\ln x)|^3}\). When \(x > e\), \(f'(x) < 0\) which implies \(f(x)\) is decreasing. Because \(f(x) = \frac{1}{x(\ln x)|\ln(\ln x)|^2}\) is positive, continuous and decreasing on \([3, \infty)\), we may apply the integral test:

\[
\int_3^\infty \frac{1}{x(\ln x)|\ln(\ln x)|^2} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{\ln(\ln x)} \right]_3^b = \frac{1}{\ln(\ln 3)} - \lim_{b \to \infty} \frac{1}{\ln(\ln b)} = \frac{1}{\ln(\ln 3)}.
\]

Thus, the series \(\sum \frac{1}{x(\ln x)|\ln(\ln x)|^2}\) converges.

(f) Since \(n^2 < n^2 + n + 1\) for all \(n \geq 0\), we have

\[
\frac{1}{n^2 + n + 1} < \frac{1}{n^2}.
\]

Because \(\sum 1/n^2\) converges (it’s a \(p\)-series with \(p = 2 > 1\)), the comparison test implies that \(\sum 1/(n^2 + n + 1)\) also converges.

(g) Since \(0 \leq n^2 - n < n^2\) and \(n^4 < n^4 + 2\) for \(n \geq 1\), we have

\[
0 \leq \frac{n^2 - n}{n^4 + 2} < \frac{n^2}{n^4} = \frac{1}{n^2}.
\]

Since the series \(\sum 1/n^2\) converges (it’s a \(p\)-series with \(p = 2 > 1\)), the comparison test implies that the series \(\sum \frac{n^2 - n}{n^4 + 2}\) also converges.
(h) Since $3^n < n + 3^n$, we have

$$\frac{n + 2^n}{n + 3^n} < \frac{n + 2^n}{3^n} = \frac{n}{3^n} + \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n .$$

Applying the ratio test to the series $\sum \frac{n}{3^n}$, we have

$$\lim_{n \to \infty} \frac{n+1}{3(n+1)} = \lim_{n \to \infty} \frac{(n+1)3^{n+1}}{n3^n} = \lim_{n \to \infty} \frac{n+1}{3n} = \frac{1}{3} < 1 .$$

Therefore, the series $\sum \frac{n}{3^n}$ converges. Since $2/3 < 1$, $\sum(2/3)^n$ is a convergent geometric series. It follows that $\sum\left(\frac{n}{3^n} + (2/3)^n\right)$ converges. Because $\sum\left(\frac{n}{3^n} + (2/3)^n\right)$ converges, the comparison test implies that $\sum\frac{n+2^n}{n+3^n}$ also converges.

(i) Since

$$\lim_{n \to \infty} \frac{n}{3n+2} = \frac{1}{3},$$

it follows that $\lim_{n \to \infty} (-1)^n \frac{n}{3n+2} \neq 0$ and hence the series $\sum (-1)^n \frac{n}{3n+2}$ diverges.

(j) Since $\sqrt{2n} < \sqrt{2n+1}$, we have

$$\frac{(-1)^n n}{\sqrt{2n+1}} \leq \frac{n}{\sqrt{2n}} = \frac{n}{2^{n/2}} .$$

Applying the ratio test to the series $\sum \frac{n}{2^{n/2}}$, we have

$$\lim_{n \to \infty} \frac{n+1}{2^{n+1/2}} = \lim_{n \to \infty} \frac{(n+1)2^{n/2}}{n2^{(n+1)/2}} = \lim_{n \to \infty} \frac{n+1}{n2^{1/2}} = \frac{1}{\sqrt{2}} < 1 ,$$

and therefore the series $\sum \frac{n}{2^{n/2}}$ converges. Finally, the absolute convergence test implies $\sum \frac{(-1)^n}{\sqrt{2}^{n+1}}$ also converges.

(k) If $n$ is a positive integer then $\cos(n\pi) = (-1)^n$. Clearly

$$0 < \frac{1}{(n+1)^{3/2}} < \frac{1}{n^{3/2}} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{(n+1)^{3/2}} = 0 .$$

Therefore, the alternating series test implies that the series $\sum \frac{\cos n\pi}{(n+1)^{3/2}}$ converges.

(l) Since $|\cos(n)| \leq 1$, we have

$$\left|\frac{(-1)^{n+1}n^2 \cos n}{3^n}\right| \leq \frac{n^2}{3^n} .$$

Applying the ratio test to the series $\sum \frac{n^2}{3^n}$ gives

$$\lim_{n \to \infty} \frac{(n+1)^2}{3(n+1)} = \lim_{n \to \infty} \frac{(n+1)^2}{n2^{n+1}} = \lim_{n \to \infty} \frac{(n+1)^2}{n2^{n+1}} = \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^2 \frac{1}{3^{2n+1}} = 0 < 1 ,$$

and therefore the series $\sum \frac{n^2}{3^n}$ converges. Hence $\sum \frac{(-1)^{n+1}n^2 \cos n}{3^n}$ converges by absolute convergence test.
(m) Applying the ratio test, we have

\[
\lim_{n \to \infty} \frac{(n+3)!}{(n+2)!} \left( \frac{(n+1)!}{3^n n!} \right)^2 = \lim_{n \to \infty} \frac{(n+3)!}{(n+2)!} \left( \frac{3^{n+1}}{(n+1)!} \right)^2 = \lim_{n \to \infty} \frac{n+3}{3(n+1)^2} = 0 < 1,
\]

and therefore the series \( \sum \frac{(n+2)!}{3^n n!} \) converges.

(n) Applying the ratio test, we have

\[
\lim_{n \to \infty} \left| \frac{(-1)^n + \frac{2\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{1\cdot 4\cdot 7\cdots(3n-2)(3n+1)}}{(-1)^{n+1} \cdot \frac{2\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 4\cdot 7\cdots(3n-2)}} \right| = \lim_{n \to \infty} \frac{2n+1}{3n+1} = \frac{2}{3} < 1,
\]

and therefore the series \( \sum (-1)^{n+1} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 4\cdot 7\cdots(3n-2)} \) converges.

12. (a) Applying the ratio test, we have

\[
\lim_{n \to \infty} \left| \frac{(n+1)2^{n+1}}{2n+1} \right| = \lim_{n \to \infty} \frac{(n+1)|x|}{2n} = \frac{|x|}{2} \lim_{n \to \infty} \frac{n+1}{n} = \frac{|x|}{2},
\]

which implies that this series converges if \( |x| < 2 \) and diverges if \( |x| > 2 \). Therefore, the radius of convergence is 2.

(b) Using the ratio test gives

\[
\lim_{n \to \infty} \left| \frac{(-4)^{n+1} x^{n+1}}{\sqrt{2n} + 3} \right| = \lim_{n \to \infty} \frac{4|x| \sqrt{2n+1}}{\sqrt{2n} + 3} = 4|x| \lim_{n \to \infty} \sqrt{\frac{2n+1}{2n+3}} = 4|x|,
\]

which implies that that this series converges if \( |x| < 1/4 \) and diverges if \( |x| > 1/4 \). Thus, the radius of convergence is 1/4.

(c) Applying the ratio test, we have

\[
\lim_{n \to \infty} \left| \frac{(2x-1)^n}{n+1} \right| = \lim_{n \to \infty} \frac{|2x-1|(n^4+16)}{(n+1)^4 + 16} = 2|x - \frac{1}{2}| \lim_{n \to \infty} \frac{n^4}{n^4} = 2|x - \frac{1}{2}|
\]

which implies that that this series converges if \( |x - 1/2| < 1/2 \) and diverges if \( |x - 1/2| > 1/2 \). Hence, the radius of convergence is 1/2.

(d) Using the ratio test yields

\[
\lim_{n \to \infty} \frac{|x(n+1)|}{|n(n+1)|} = \lim_{n \to \infty} \frac{|x|n^n}{(n+1)^n} = |x| \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n.
\]
Now, if \( f(n) = \left( \frac{n}{n+1} \right)^n \) then

\[
\lim_{n \to \infty} \ln f(n) = \lim_{n \to \infty} n \ln \left( \frac{n}{n+1} \right) = \lim_{n \to \infty} \frac{\ln \left( \frac{n}{n+1} \right)}{n} = \lim_{n \to \infty} -\frac{n+1-n}{(n+1)^2} = \lim_{n \to \infty} -\frac{1}{n^2} = -1.
\]

Hence, \( \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \lim_{n \to \infty} e^{\ln f(n)} = e^{-1} \). Returning to the ratio test, we see that

\[
\lim_{n \to \infty} \left| \frac{f(n+1)x^{n+1}}{f(n)x^n} \right| = e^{-1} |x|,
\]

which implies that this series converges if \(|x| < e\) and diverges if \(|x| > e\). Thus, the radius of convergence is \( e \).

(e) Applying the ratio test, we have

\[
\lim_{n \to \infty} \left| \frac{(n+1)!x^{n+1}}{n!} \right| = \lim_{n \to \infty} \frac{|x-2|(n+1)^2}{n^2} = |x-2| \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^2 = |x-2|,
\]

which implies that this series converges if \(|x-2| < 1\) and diverges if \(|x-2| > 1\). Therefore, the radius of convergence is 1.

(f) Using the ratio test, we obtain

\[
\lim_{n \to \infty} \left| \ln(n+1)x^{n+1} \right| = \lim_{n \to \infty} \frac{|x| \ln(n+1)}{3 \ln(n)} = \frac{1}{3} |x| \lim_{n \to \infty} \frac{n+1}{n} = \frac{1}{3} |x|,
\]

which implies that this series converges if \(|x| < 3\) and diverges if \(|x| > 3\). Thus, the radius of convergence is 3.

(g) Applying the ratio test gives

\[
\lim_{n \to \infty} \left| \frac{n+10^n(x-10)^{2n+2}}{(n+1)!} \right| = \lim_{n \to \infty} \frac{10|x-10|^2}{n+1} = 10|x-10|^2 \lim_{n \to \infty} \frac{1}{n+1} = 0,
\]

which implies that this series converges for all \( x \). Therefore, the radius of convergence is \( \infty \).

(h) Using the ratio test implies

\[
\lim_{n \to \infty} \left| \frac{x^2+1}{5} \right|^{n+1} = \lim_{n \to \infty} \frac{x^2+1}{5},
\]
which implies that this series converges if
\[ x^2 + 1 < 5 \quad \Rightarrow \quad |x| < 2. \]
Similarly, the series diverges if \(|x| > 2\). Therefore, the radius of convergence is 2.

13. (a) Since
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \]
we have \[1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \cdots + \frac{2^n}{n!} + \cdots = e^2.\]
(b) Since
\[ \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \]
we have \[1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots = \sin(1).\]
(c) Since
\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \]
we have \[1 + \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots + \frac{1}{4^n} + \cdots = \frac{1}{1-1/4} = \frac{4}{3}.\]
(d) Since
\[ \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \]
we have \[1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots = \cos(1) = \cos(-1).\]

14. The second degree Taylor polynomial for a function \(f\) about \(x = 0\) is given by
\[ f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2. \]
In particular, \(a = f(0)\), \(b = f'(0)\) and \(2c = f''(0)\).
(a) Since the y-intercept is positive, \(a > 0\); since \(f(x)\) is decreasing at \(x = 0\), \(b\) is negative; because \(f(x)\) is concave-down at \(x = 0\), \(c\) is also negative.
(b) Since the y-intercept is negative, \(a < 0\); since \(f(x)\) is increasing at \(x = 0\), \(b\) is positive; because \(f(x)\) is concave-up at \(x = 0\), \(c\) is also positive.

15. The fourth degree Taylor approximation for \(e^h\) for \(h\) near 0 is
\[ P_4(h) = 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!}. \]
(a) Evaluating the limit, we have
\[ \lim_{h \to 0} \frac{e^h - 1 - h}{h^2} = \lim_{h \to 0} \frac{1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!} - 1 - h}{h^2} = \lim_{h \to 0} \frac{\frac{1}{2} + \frac{h}{3!} + \frac{h^2}{4!}}{\frac{1}{2}} = \frac{1}{2}. \]
Clearly, any Taylor approximation of degree at least two will give the same answer.

(b) Evaluating the limit, we have
\[
\lim_{h \to 0} \frac{e^h - 1 - h - \frac{h^2}{2}}{h^3} = \lim_{h \to 0} \frac{1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!} - 1 - h - \frac{h^2}{2}}{h^3} = \lim_{h \to 0} \frac{1}{3!} + \frac{h}{4!} = \frac{1}{6}.
\]

Clearly, any Taylor approximation of degree at least three will give the same answer.

16. (a) Using the binomial series, we have
\[
\sqrt{1 + y} = 1 + \frac{(\frac{1}{2})}{1!} y + \frac{(\frac{1}{2})(\frac{1}{2} - 1)}{2!} y^2 + \frac{(\frac{1}{2})(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!} y^3 + \cdots
\]
\[
= 1 + \frac{1}{2} y - \frac{1}{8} y^2 + \frac{1}{16} y^3 + \cdots
\]

Setting \(y = -2x\) gives \(\sqrt{1 - 2x} = 1 - x - \frac{1}{2} x^2 - \frac{1}{2} x^3 + \cdots\).

(b) The geometric series is \(\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots\). Integrating both sides of this equation with respect to \(x\) gives
\[
-\ln(1 - x) = C + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \cdots
\]
Since \(\ln(1) = 0\), it follows that \(C = 0\). Finally, setting \(x = 2y\), we have
\[
\ln(1 - 2y) = -2y - 2y^2 - \frac{8}{3} y^3 - 4y^4 - \cdots
\]

(c) Since \(e^y = 1 + y + \frac{1}{2!} y^2 + \frac{1}{3!} y^3 + \frac{1}{24} y^4 + \cdots\), setting \(y = -z^2\) gives
\[
e^{-z^2} = \frac{1}{e^{z^2}} = 1 - z^2 + \frac{1}{2} z^4 - \frac{1}{6} z^6 + \frac{1}{24} z^8 + \cdots
\]

Multiplying both sides of this equation by \(z\), we obtain
\[
\frac{z}{e^{z^2}} = z - z^3 + \frac{1}{2} z^5 - \frac{1}{6} z^7 + \frac{1}{24} z^9 + \cdots
\]

(d) Recall that \(\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\) and the binomial series implies
\[
\sqrt{1 + t} = 1 + \frac{(\frac{1}{2})}{1!} t + \frac{(\frac{1}{2})(\frac{1}{2} - 1)}{2!} t^2 + \frac{(\frac{1}{2})(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!} t^3 + \cdots
\]
\[
= 1 + \frac{1}{2} t - \frac{1}{8} t^2 + \frac{1}{16} t^3 + \cdots
\]

Multiplying these two power series gives:
\[
\sqrt{1 + t} \sin(t) = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \left(1 + \frac{1}{2} t - \frac{1}{8} t^2 + \frac{1}{16} t^3 + \cdots\right)
\]
\[
= t + \frac{1}{2} t^2 - \frac{7}{24} t^3 - \frac{1}{48} t^4 - \cdots
\]
17. (a) The approximation \( \sin(\theta) \approx \theta \) comes from the linear approximation to \( \sin(\theta) \) at \( \theta = 0 \). When the graph of \( \sin(\theta) \) is below that of \( \theta \) the approximation is an overestimate and this occurs on the interval \( 0 < \theta < 1 \). Similarly, when the graph of \( \sin(\theta) \) is above \( \theta \) the approximation is an underestimate and this occurs on the interval \( -1 < \theta < 0 \).

(b) Recall that the following: Suppose \( f \) and all its derivatives are continuous. If \( P_n(x) \) is the \( n \)-th Taylor approximation to \( f(x) \) about \( a \), then

\[
|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1},
\]

where \( M \) is the maximum value of \( |f^{(n+1)}(x)| \) on the interval between \( a \) and \( x \). Now, \( \sin \theta \approx \theta \) is the 2-nd degree Taylor approximation to \( \sin \theta \) about 0. Since the third derivative of \( \sin \theta \) is \( -\cos \theta \) which is bounded above by 1, we have

\[
|\sin \theta - \theta| \leq \frac{1}{3!}|\theta|^3 \leq \frac{1}{6},
\]
on the interval \([-1, 1]\).

18. Recall that \( \frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \). Using the binomial expansion, we have

\[
\frac{1}{\sqrt{1+y}} = (1+y)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!}y^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!}y^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n(1\cdot3\cdot5\cdot7\cdots(2n-1))}{2^nn!}y^n.
\]

Setting \( y = -x^2 \), we obtain

\[
\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2^nn!}x^{2n}.
\]

Integrating both sides of this equation with respect to \( x \) yields

\[
\arcsin(x) + C = \sum_{n=0}^{\infty} \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2^nn!(2n+1)}x^{2n+1}.
\]

Since \( \arcsin(0) = 0 \), we have

\[
\arcsin(x) = \sum_{n=0}^{\infty} \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2^nn!(2n+1)}x^{2n+1}.
\]
19. The derivatives of (i) – (iii) are:

(i) \[ \frac{dy}{dx} = -\frac{A}{x^2} + 2Bx \] and \[ \frac{d^2y}{dx^2} = \frac{2A}{x^3} + 2B = \frac{2}{x^2} \left( \frac{A}{x^2} + bX^2 \right) \]

(ii) \[ \frac{dy}{dx} = A \cos(x) - B \sin(x) \] and \[ \frac{d^2y}{dx^2} = -A \sin(x) - B \cos(x) \]

(iii) \[ \frac{dy}{dx} = \frac{1}{\sqrt{2x + A}} \] and \[ \frac{d^2y}{dx^2} = -(2x + A)^{-3/2} \]

Hence, (i) is a solution to (c), (ii) is a solution to (a), and (iii) is a solution to (b).

20. (a) (i) Euler’s method for \( y' = (\sin x)(\sin y) \) starting at \( (0, 2) \)

\[
\begin{array}{c|c|c}
 x & \text{approximate } y & \Delta y = (\text{slope})\Delta x \\
\hline
0 & 2 & 0 = \sin(0) \cdot \sin(2) \cdot (0.1) \\
0.1 & 2 & 0.009 = \sin(0.1) \cdot \sin(2) \cdot (0.1) \\
0.2 & 2.009 & 0.018 = \sin(0.2) \cdot \sin(2.009) \cdot (0.1) \\
0.3 & 2.027 & 0.027 = \sin(0.3) \cdot \sin(2.027) \cdot (0.1) \\
0.4 & 2.054 & \\
\end{array}
\]

(ii) Euler’s method for \( y' = (\sin x)(\sin y) \) starting at \( (0, \pi) \)

\[
\begin{array}{c|c|c}
 x & \pi & \Delta y = (\text{slope})\Delta x \\
\hline
0 & \pi & 0 = \sin(0) \cdot \sin(\pi) \cdot (0.1) \\
0.1 & \pi & 0 = \sin(0.1) \cdot \cos(\pi) \cdot (0.1) \\
0.2 & \pi & 0 = \sin(0.2) \cdot \cos(\pi) \cdot (0.1) \\
0.3 & \pi & 0 = \sin(0.3) \cdot \cos(\pi) \cdot (0.1) \\
0.4 & \pi & \\
\end{array}
\]

(b) From the slope field, we see that solution curve passing through \( (0, 2) \) is increasing and that \( y = \pi \) is an equilibrium solution to the differential equation.

21. (a) Separating variables, we obtain

\[
\int e^{-z} \, dz = \int t \, dt \quad \Rightarrow \quad -e^{-z} = \frac{1}{2}t^2 + C \quad \Rightarrow \quad z = -\ln \left( -\frac{1}{2}t^2 - C \right).
\]

Since \( z(0) = 0 \), it follows that \(-\frac{1}{2}(0)^2 - C = 1 \) and \( C = -1 \). Thus, the solution to the initial-value problem is \( z(t) = -\ln \left( -\frac{1}{2}t^2 - 1 \right) = \ln \left( \frac{2}{2-t^2} \right) \).

(b) Separating variables yields

\[
\int \frac{dy}{y^2} = \int 1 + t \, dt \quad \Rightarrow \quad -\frac{1}{y} = t + \frac{1}{2}t^2 + K \quad \Rightarrow \quad y = -\frac{1}{t + \frac{1}{2}t^2 + K}.
\]

Since \( y(1) = 2 \), we have \( 1 + \frac{1}{2} + K = -\frac{1}{2} \) and \( K = -2 \). Hence, the solution to the initial-value problem is \( y(t) = -\frac{1}{t + \frac{1}{2}t^2 - 2} = \frac{-2}{t^2 + 2t - 4} \).
Figure 1. Slope field for $y' = (\sin x)(\sin y)$.

(c) Separating variables gives

$$\int \frac{dw}{w^2} = \int \theta \sin(\theta^2) \, d\theta \implies -\frac{1}{w} = -\frac{1}{2} \cos(\theta^2) + C \implies w = \frac{1}{\frac{1}{2} \cos(\theta^2) - C}.$$ 

Since $w(0) = 1$, we have $\frac{1}{2} - C = 1$ and $C = -\frac{1}{2}$. Therefore, the solution to the initial-value problem is $w(\theta) = \frac{1}{\frac{1}{2} \cos(\theta^2) + \frac{1}{2}} = \cos(\theta^2) + 1$.

(d) Separating variables, we have

$$\int \frac{du}{u^2} = \int \frac{dx}{x(x+1)} = \int \frac{1}{x} - \frac{1}{x+1} \, dx$$

$$\implies -\frac{1}{u} = \ln |x| - \ln |x+1| + K \implies u = \frac{1}{\ln \left|\frac{x+1}{x}\right| - K}.$$ 

Since $u(1) = 1$, it follows that $K = \ln(2) - 1$. Hence, the solution to the initial-value problem is $u(x) = \frac{1}{\ln \left|\frac{x+1}{x}\right| + 1 - \ln(2)} = \frac{1}{\ln \left|\frac{x+1}{2x}\right| + 1}$.

22. (a) Separating variables, we have

$$\int \frac{dR}{R} = \int k \, dt \implies \ln |R| = kt + C \implies R = R_0 e^{kt},$$

where $R_0 = \pm e^C$. Thus, the general solution to the differential equation is $R(t) = R_0 e^{kt}$. 
(b) Separating variables gives
\[ \int \frac{dP}{b-aP} = \int dt \quad \Rightarrow \quad \ln |b + aP| = t + C \quad \Rightarrow \quad P = \frac{1}{a} (Ke^t - b), \]
where \( K = \pm e^C \). Hence, the general solution to the differential equation is \( P(t) = \frac{1}{a} (Ke^t - b) \).

(c) Separating variables, we obtain
\[ \int \cos x \sin x \, dx = \int \frac{1+2 \ln t}{t} \, dt \quad \Rightarrow \quad \ln |\sin x| = \ln |t| + (\ln |t|)^2 + K \quad \Rightarrow \quad \sin(x) = Cte^{(\ln t)^2}, \]
where \( C = \pm e^C \). Hence, the general solution to the differential equation is \( x(t) = \arcsin \left( Cte^{(\ln t)^2} \right) \).

(d) Separating variables gives:
\[ \int \frac{dx}{\ln(x)} = \int \frac{dt}{t} \quad \Rightarrow \quad \ln |\ln x| = \ln |t| + C \quad \Rightarrow \quad \ln x = Kt \quad \Rightarrow \quad x = e^{Kt}, \]
where \( K = \pm e^C \). Therefore, the general solution to the differential equation is \( x(t) = e^{Kt} \).

23. (a) Since the rate of growth of a tumor is proportional to the size of the tumor, we have
\[ \frac{dS}{dt} = kS, \]
where \( k \) is a constant.

(b) Separating variables, we have
\[ \int \frac{dS}{S} = \int k \, dt \quad \Rightarrow \quad \ln |S| = kt + C \quad \Rightarrow \quad S = S_0 e^{kt}, \]
where \( S_0 = \pm e^C \). Therefore the general solution is \( S(t) = S_0 e^{kt} \).

(c) Since \( S(0) = 5 \), it follows that \( S_0 = 5 \) and \( S(t) = 5e^{kt} \) is the solution to the initial value problem.

(d) Since \( S(3) = 8 \), we have \( 8 = 5e^{3k} \) which implies \( k = \frac{1}{3} \ln \left( \frac{8}{5} \right) \approx 0.157 \). Hence, the solution is \( S(t) = 5 \left( \frac{8}{5} \right)^{t/3} \).

24. (a) The volume \( V \) of a spherical snowball is \( V = \frac{4}{3} \pi r^3 \) where \( r \) is the radius. Solving for \( r \) gives \( r = \left( \frac{4V}{4\pi} \right)^{1/3} \). On the other hand, the surface area \( S \) of a sphere is \( S = 4\pi r^2 \) and substitution yields \( S = 4\pi \left( \frac{3V}{4\pi} \right)^{2/3} \). Therefore, the differential equation for the volume is
\[ \frac{dV}{dt} = k4\pi \left( \frac{3V}{4\pi} \right)^{2/3}, \]
where $k$ is a constant. For simplicity, we combine the constants and write $\frac{dV}{dt} = AV^{\frac{2}{3}}$, where $A$ is a constant. Because $V$ is decreasing, $A$ will be negative.

(b) Separating variables, we have
\[
\int V^{-\frac{2}{3}} \, dV = \int A \, dt \quad \Rightarrow \quad 3V^{\frac{1}{3}} = At + C \quad \Rightarrow \quad V = \left(\frac{At+C}{3}\right)^3.
\]
Since $V(0) = V_0$, we have $V_0 = \left(\frac{C}{3}\right)^3$ and $C = 2V_0^{\frac{1}{3}}$. Thus, the general solution is $V(t) = \left(\frac{4}{3}t + V_0^{\frac{1}{3}}\right)^3$.

(c) The snowball disappears when $V(t) = 0$. Solving for $t$ gives
\[
\left(\frac{4}{3}t + V_0^{\frac{1}{3}}\right)^3 = 0 \quad \Rightarrow \quad t = -\frac{3}{4}V_0^{\frac{1}{3}};
\]
note that $t$ is positive because $a$ is negative.