The final exam is cumulative. Your studying time will be best spent if you work carefully through this review sheet and go back over previous homeworks and exams.

1. Evaluate the following integrals.

   (a) \[ \int \frac{1}{\sqrt{4-x}} \, dx \]
   (b) \[ \int \frac{e^t - e^{-t}}{e^t + e^{-t}} \, dt \]
   (c) \[ \int \frac{z \cos(z^2)}{\sqrt{\sin(z^2)}} \, dz \]
   (d) \[ \int (\ln w)^2 \, dw \]
   (e) \[ \int (y + 2)\sqrt{2 + 3y} \, dy \]
   (f) \[ \int \frac{1}{x^2 + 6x + 25} \, dx \]
   (g) \[ \int w^2 \sin(3w) \, dw \]
   (h) \[ \int \frac{z - 1}{\sqrt{2z - z^2}} \, dz \]
   (i) \[ \int t^3 \ln t \, dt \]
   (j) \[ \int \frac{y^4 + 3y^3 + 2y^2 + 1}{y^2 + 3y + 2} \, dy \]
   (k) \[ \int \frac{e^t \, dt}{e^{2t} + 3e^t + 2} \]
   (l) \[ \int \frac{\sin \theta \, d\theta}{\cos^2 \theta + \cos \theta - 2} \]
   (m) \[ \int \frac{2x^3 - 2x^2 + 1}{x^2 - x} \, dx \]
   (n) \[ \int \frac{2s + 2}{(s^2 + 1)(s - 1)^2} \, ds \]
   (o) \[ \int_1^\infty \frac{3z^2 + z + 4}{z^3 + z} \, dz \]
   (p) \[ \int_{-1}^0 \frac{x^3 \, dx}{x^2 - 2x + 1} \]
   (q) \[ \int \frac{1 - w}{\sqrt{1 - w^2}} \, dw \]
   (r) \[ \int_0^{1/2} \frac{2 - 8y}{1 + 4y^2} \, dy \]
   (s) \[ \int \frac{dt}{\sqrt{-t^2 + 4t - 3}} \]
   (t) \[ \int \arctan(x) \, dx \]
   (u) \[ \int \frac{4 \, dw}{1 + (2w + 1)^2} \]
   (v) \[ \int_1 \frac{e^{\pi/3} \, dz}{z \cos(\ln(z))} \]
   (w) \[ \int \frac{6 \, dy}{\sqrt{y(1 + y)}} \]
   (x) \[ \int \frac{(1 + \ln t) \, dt}{t\sqrt{(\ln t)(2 + \ln t)}} \]
   (y) \[ \int (27)^{29+1} \, d\theta \]
   (z) \[ \int \sin(2x)e^{\cos(2x)} \, dx \]
2. Derive the following formulas:

\( \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx \)

(b) \( \int x^n \cos(ax) \, dx = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, dx \)

(c) \( \int \cos^n(x) \, dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x) \, dx \)

3. Calculate the following integrals, if they converge.

\( \int_{1}^{\infty} \frac{1}{5x + 2} \, dx \)

(b) \( \int_{0}^{\infty} te^{-t} \, dt \)

(c) \( \int_{-\infty}^{\infty} \frac{dz}{z^2 + 25} \)

(d) \( \int_{2}^{\infty} \frac{dw}{w \ln w} \)

(e) \( \int_{1}^{2} \frac{dw}{w \ln w} \)

(f) \( \int_{0}^{1} \frac{1}{\sqrt{4 - x^2}} \, dx \)

(g) \( \int_{0}^{\pi} \frac{1}{\sqrt{y}} \, e^{-\sqrt{y}} \, dy \)

(h) \( \int_{3}^{6} \frac{d\theta}{(4 - \theta)^2} \)

4. Find the volume of the following solids. The base of a solid is the region between the curve \( y = 2\sqrt{\sin x} \) and the interval \( 0 \leq x \leq \pi \) on the x-axis. The cross sections perpendicular to the x-axis are

(a) equilateral triangles with bases running from the x-axis to the curve.

(b) squares with bases running from the x-axis to the curve.

5. Find the volumes of the solids generated by revolving the following regions about the x-axis.

(a) \( y = x^2, \ y = 0, \ x = 2; \)

(b) \( y = \sqrt{9 - x^2}, \ y = 0; \)

(c) \( y = e^{-x}, \ y = 0, \ x = 0, \ x = 1. \)

6. For each of the following functions, pick which approximations — left, right, trapezoid or midpoint — is guaranteed to give an overestimate for \( \int_{0}^{5} f(x) \, dx \) and which is guaranteed to give an underestimate. (There may be more than one).

7. Find the length of the following curves.

(a) \( y = (1/3)(x^2 + 2)^{3/2} \) from \( x = 0 \) to \( x = 3; \)

(b) \( x = \frac{1}{2}(e^y + e^{-y}) \) from \( y = -1 \) to \( y = 1; \)

(c) \( y = \frac{x^3}{4} + \frac{1}{8x^2} \) from \( x = 1 \) to \( x = 2. \)

(d) \( y = \ln(1 - x^2) \) from \( x = 0 \) to \( x = 1/2. \)
8. While taking a walk along the road where you live, you accidentally drop your glove. You don’t know where you dropped it. Suppose the probability density $p(x)$ for having dropped the glove $x$ kilometers from home (along the road) is
\[ p(x) = 2e^{-2x} \quad \text{for} \quad x \geq 0. \]

(a) What is the probability that you dropped it within 1 kilometer of home?
(b) At what distance $y$ from home is the probability that you dropped it with $y$ kilometers equal to 0.95?

9. Cephalexin is an antibiotic taken in tablets of 250 mg every six hours. It is known that at end of six hours, about 1% of the drug is still in the body.

(a) Write an expression for $Q_1$, $Q_2$, $Q_3$ where $Q_n$ is the amount of cephalexin, measured in mg, in the body right after the $n$-th tablet is taken.
(b) Write an expression for $Q_n$ and put it is closed-form.
(c) If the patient keeps taking the tablets, use your answer to part (c) to find the quantity of cephalexin in the body in the long run, right after taking a pill.

10. Find the sum of the infinite series
\[ 1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \frac{2}{10^7} + \frac{3}{10^8} + \frac{7}{10^9} + \cdots. \]

11. Determine whether the following infinite series converges or diverges

(a) \[ \sum_{n=1}^{\infty} (\sqrt{2})^{1-n} \]
(b) \[ \sum_{n=0}^{\infty} \frac{1+2^n+3^n}{5^n} \]
(c) \[ \sum_{n=1}^{\infty} ne^{-n^2} \]
(d) \[ \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \]
(e) \[ \sum_{n=3}^{\infty} \frac{1}{n(\ln n)[\ln(\ln n)]^2} \]
(f) \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1} \]
(g) \[ \sum_{n=1}^{\infty} \frac{n^2 - n}{n^4 + 2} \]
(h) \[ \sum_{n=1}^{\infty} \frac{n + 2^n}{n + 3^n} \]
(i) \[ \sum_{n=1}^{\infty} \frac{(-1)^n n}{3n + 2} \]
(j) \[ \sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{2^n + 1}} \]
(k) \[ \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/2}} \]
(l) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 \cos n}{3n^4} \]
(m) \[ \sum_{n=1}^{\infty} \frac{(n + 2)!}{3^n (n!)^2} \]
(n) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n - 1)}{1 \cdot 4 \cdot 7 \cdots (3n - 2)} \]
12. Find the radius of convergence of each of the following power series.

(a) \( \sum_{n=1}^{\infty} \frac{nx^n}{2^n} \)

(b) \( \sum_{n=0}^{\infty} \frac{(-4)^nx^n}{\sqrt{2n+1}} \)

(c) \( \sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^4 + 16} \)

(d) \( \sum_{n=1}^{\infty} \frac{n!x^n}{n^n} \)

(e) \( \sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{n^2} \)

(f) \( \sum_{n=1}^{\infty} \frac{(\ln n)x^n}{3^n} \)

(g) \( \sum_{n=1}^{\infty} \frac{(-1)^n+10^n(x-10)^{2n}}{n!} \)

(h) \( \sum_{n=0}^{\infty} \frac{(x^2 + 1)^n}{5} \)

13. By recognizing each of the following series as a Taylor series evaluated at a particular value of \( x \), find the sum of each of the following convergent series.

(a) \( 1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \cdots + \frac{2^n}{n!} + \cdots \)

(b) \( 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots \)

(c) \( 1 + \frac{1}{2!} + \frac{1}{10} - \frac{1}{31} + \cdots + \frac{1}{15} + \cdots \)

(d) \( 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \)

14. Suppose \( p(x) = a + bx + cx^2 \) is the second degree Taylor polynomial for the function \( f \) about \( x = 0 \). What can you say about the signs of \( a \), \( b \), \( c \) if \( f \) has the graph given below?

15. Use the fourth degree Taylor approximation for \( e^h \), for \( h \) near 0, to evaluate the following limits. Would your answer be different if you used a Taylor polynomial of higher degree?

(a) \( \lim_{h \to 0} \frac{e^h - 1 - h}{h^2} \)

(b) \( \lim_{h \to 0} \frac{e^h - 1 - h - \frac{h^2}{2}}{h^3} \)

16. Find the first four nonzero terms of the Taylor series about 0 for each of the following functions.

(a) \( \sqrt{1 - 2x} \)

(b) \( \ln(1 - 2y) \)

(c) \( \frac{z}{e^{z^2}} \)

(d) \( \sqrt{1 + t \sin t} \)

17. Consider the error in using the approximation \( \sin \theta \approx \theta \) on the interval \([-1, 1] \).
(a) Reasoning informally, say where the approximation is an overestimate and where it is an underestimate.
(b) Estimate the magnitude of the largest possible error.

18. Find the Taylor series expansion of arcsin $x$.

19. Pick out which functions are solutions to which differential equations. Assume $A$ and $B$ are constants.

   (a) $\frac{d^2y}{dx^2} = -y$  \hspace{1cm} (i) \hspace{1cm} $y = \frac{A}{x} + Bx^2$

   (b) $\frac{dy}{dx} = \frac{1}{y}$  \hspace{1cm} (ii) \hspace{1cm} $y = A \sin(x) + B \cos(x)$

   (c) $x^2 \frac{d^2y}{dx^2} = 2y$  \hspace{1cm} (iii) \hspace{1cm} $y = \sqrt{2x + A}$

20. Consider the differential equation $y' = (\sin x)(\sin y)$.
   (a) Calculate approximate $y$-values using Euler’s method and $\Delta x = 0.1$, starting at each of the following points: (i) $(0, 2)$; (ii) $(0, \pi)$.
   (b) Plot the slope field for this differential equation and use it to explain your solution to part (a)(ii).
21. Find the solutions to the following differential equations subject to the given initial conditions.

(a) \[ \frac{dz}{dt} = te^z, \quad \text{through the origin.} \]

(b) \[ \frac{dy}{dt} = y^2(1 + t), \quad y = 2 \text{ when } t = 1. \]

(c) \[ \frac{dw}{d\theta} = \theta w^2 \sin(\theta^2), \quad w(0) = 1. \]

(d) \[ x(x+1)\frac{du}{dx} = u^2, \quad u(1) = 1. \]

22. Solve the following differential equations.

(a) \[ \frac{dR}{dt} = kR, \quad \text{where } k \text{ is a constant.} \]

(b) \[ \frac{dP}{dt} - aP = b, \quad \text{where } a \text{ and } b \text{ are constants.} \]

(c) \[ t \frac{dx}{dt} = (1 + 2 \ln t) \tan x, \quad \text{where } x \geq 0. \]

(d) \[ \frac{dx}{dt} = \frac{x \ln x}{t}, \quad \text{where } x \geq 0. \]

23. The rate of growth of a tumor is proportional to the size of the tumor.

(a) Write a differential equation satisfied by \( S \), the size of the tumor in millimeters, as a function of time, \( t \).

(b) Find a general solution to the differential equation.

(c) If the tumor is 5 mm across at time \( t = 0 \), what does that tell you about the solution?

(d) If, in addition, the tumor is 8 mm across at time \( t = 3 \), what does that tell you about the solution?

24. A spherical snowball melts at a rate proportional to its surface area.

(a) Write a differential equation for its volume \( V \).

(b) If the initial volume is \( V_0 \), solve the differential equation.

(c) When does the snowball disappear?