1. Suppose \( F = 2xyi + (x^2 + byz)j + y^2k \).

   A. For what number \( b \) does \( \text{div}(F) = \text{curl}(F) \cdot i \)?
   
   Note that \( \text{div}(F) = 2y + bz \) and \( \text{curl}(F) = (2y - by)i + 0j + 0k \). So \( \text{div}(F) = \text{curl}(F) \cdot i \) when \( b = 0 \).

   B. For what (different) number \( b \) is \( F \) conservative? Since \( F \) is conservative when \( \text{curl}(F) = 0 \) we need \( b = 2 \).

   C. For this second number \( b \), find \( f \) so that \( \nabla(f) = F \). One choice is \( f = x^2y + y^2z \).

2. A. Rewrite the following integral as an iterated integral in the order \( dydxdz \):

   \[
   \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z) \, dz \, dy \, dx.
   \]

   The answer is

   \[
   \int_{0}^{1} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1} f(x, y, z) \, dy \, dx \, dz.
   \]

   B. Rewrite the integral in spherical coordinates.

   \[
   \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z \, dz \, dy \, dx.
   \]

   The answer is

   \[
   \int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{2} \rho^3 \cos(\phi) \sin(\phi) \, d\rho \, d\phi \, d\theta.
   \]

C. Evaluate \( \int \int_{R} e^{9x^2+4y^2} \, dA \) where \( R \) is the region bounded by \( 9x^2+4y^2 = 1 \).

Let \( x = u/3 \) and \( y = v/2 \). Let \( S \) be the region in the \((u,v)\)-plane bounded by \( u^2 + v^2 = 1 \). Then the Jacobian of this transformation is \( J = 1/6 \). So the answer is

\[
\int_{S} \int 1/6 e^{u^2+v^2} \, dA = \int_{0}^{2\pi} \int_{0}^{1} 1/6 \, e^{r^2} \, r \, dr \, d\theta = e \pi /6.
\]
3. A. Let \( F = (1 + \tan(x), x^2 + e^y) \) be a force field. Let \( C \) be the boundary of the region enclosed by the parabola \( x = y^2 \) and the lines \( x = 1 \) and \( y = 0 \). Find the work done by \( F \) as a particle travels once around \( C \) in the counterclockwise direction.

The work is \( W = \int_C F \cdot dr = \int_C (1 + \tan(x))dx + (x^2 + e^y)dy \). By Green’s Theorem, \( W = \int_0^1 \int_{y^2}^1 2x dx dy = \int_0^1 (1 - y^4)dy = 4/5 \).

B. (5 points) Find a vector field \( F \) such that \( \int_C F \cdot dr = 0 \) whenever the endpoints of \( C \) both lie on the curve \( y = x^3 + x + 1 \).

If \( f(x,y) = y - x^3 - x - 1 \), then \( F = \nabla(f) = (-3x^2 - 1)i + j \). Suppose \( C \) starts at \( a = (x_1, y_1) \) and ends at \( b = (x_2, y_2) \) where \( a \) and \( b \) lie on this curve and thus \( f(a) = f(b) = 0 \). By the Fundamental Theorem of line integrals, \( \int_C F \cdot dr = f(b) - f(a) = 0 - 0 \).

4. Consider the surface \( S \) in \( \mathbb{R}^3 \) given parametrically by \( x = u \cos(v), y = u \sin(v), \) and \( z = u \). Let \( (u,v) \) range through the domain \( D = \{(u,v) | 0 \leq u \leq 1, 0 \leq v \leq 2\pi \} \).

A. Graph \( S \). Mark the grid curves \( u = 1 \) and \( v = 0 \).

See picture \( V \) on page 1091 for a picture of this cone. The curve \( u = 1 \) is a circle and \( v = 1 \) is a line.

B. Find the surface area (for \( (u,v) \in D \)).

Let \( r(u,v) = (u \cos(v), u \sin(v), u) \). Then \( r_u = (\cos(v), \sin(v), 1) \) and \( r_v = (-u \sin(v), u \cos(v), 0) \). Thus \( r_u \times r_v = (-u \cos(v), -u \sin(v), u) \).

Thus \( |r_u \times r_v| = \sqrt{2}u \). The surface area is \( SA = \int_0^1 \int_0^{2\pi} \sqrt{2} u dv du \). Thus \( SA = (2\pi) \sqrt{2}/2 = \sqrt{2}\pi \).

C. Let \( C \) be the grid curve \( v = 0, 0 \leq u \leq 1 \). Find \( \int_C 1 ds \). What physical quantity does this integral represent?

Here \( r(u,0) = (u,0,u) \) and \( r'(u) = (1,0,1) \) for \( 0 \leq u \leq 1 \). Note \( r'(u) \) has length \( \sqrt{2} \). By page 1053, \( \int_C 1 ds = \int_0^1 \sqrt{2} du = \sqrt{2} \) is the arclength.