Mathematics V1205y, Calculus IIIS/IVA

Final: May 9, 2001, 9:00 am –12:00 noon.

The following final has 20 problems which are each worth 5 points. Please read the exam carefully and check all your answers. Show all your work. Some of the problems are harder than others, so move on if you get stuck.

**Multiple Choice:** No partial credit. Circle exactly one answer for each question.

1. Name:_____________________

2. A hummingbird is hovering at the point (2, 4, 2) and the surrounding temperature is given by the function \( f(x, y, z) = \sqrt{xyz} \). Find the rate of change of the temperature as the hummingbird flies in the direction of the vector \( \vec{v} = \langle 4, 2, -4 \rangle \).
   a. 1/2  b. 1/4  c. 1/6  d. 1  e. 3/2

3. Icecream fills the space below the sphere \( x^2 + y^2 + z^2 = a^2 \) and above the cone \( z = \sqrt{x^2 + y^2} \). The density of the icecream is given by the function \( f(x, y, z) = z \).
   Find the mass of the icecream using spherical coordinates.
   a. \( \pi a^4/8 \)  b. \( \pi a^2/\sqrt{2} \)  c. \( 4\pi a^3/3 \)  d. \( 2\pi(1 - \frac{\sqrt{2}}{2})a^3/3 \)  e. \( \pi a^4/4 \)
4. Which of the following equations is a parametric representation for this surface?
   a. \( x = 2 + \cos(u), \ y = 2 + \sin(u), \ z = u + \sin(v) \)
   b. \( x = (2 + \sin(v)) \cos(u), \ y = (2 + \sin(v)) \sin(u), \ z = u + \cos(v) \)
   c. \( x = u + \cos(v), \ y = (2 + \sin(v)) \cos(u), \ z = (2 + \sin(v)) \sin(u) \)
   d. \( x = \sin(v) \cos(u), \ y = \sin(v) \sin(u), \ z = u \)
   e. \( x = \cos(u), \ y = \sin(u), \ z = v \)

5. Suppose \( F(x, y, z) = x \mathbf{k} \) and \( S \) is the part of the plane \( z = x \) lying over the square \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \). Find the flux of \( F \) across \( S \).
   a. 2  b. \( \sqrt{2}/2 \)  c. \( \sqrt{3}/2 \)  d. 1/2  e. 1/4
6. A direction field is given below. Which of the following represents its differential equation?
   a. \( \frac{dy}{dx} = y^2 - 1 \)
   b. \( \frac{dy}{dx} = y - x \)
   c. \( \frac{dy}{dx} = x^2 - y^2 \)
   d. \( \frac{dy}{dx} = y^2 - x^2 \)
   e. \( \frac{dy}{dx} = \sin(y - x) \)

7. A fishtank contains 100 L of water with 7 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 10 L/min. How much salt is in the tank after 6 minutes?
   a. \( e^{-6} + 6 \)  b. \( 7e^{-6} \) kg  c. \( 7e^6 \) kg  d. \( 6e^{-7} \) kg  e. \( 7e^{-0.06} \) kg
8. In the power series solution $\sum_{n=0}^{\infty} a_n x^n$ of the differential equation $y'' - xy' - y = 0$, what recursion formula do the coefficients $a_n$ satisfy for $n > 0$?
   a. $a_{n+2} = a_n/(n + 2)$  b. $a_{n+2} = a_n/n(n + 1)$  c. $a_{n+2} = a_n/(n + 1)$
   d. $a_{n+1} = a_n/n(n + 1)$  e. $a_{n+1} = a_n/(n + 1)$

9. If $a + bi = (\sqrt{3} + i)^{11}$, find the pair $(a, b)$.
   a. $(a, b) = (\sqrt{3}^{11}, 1)$  b. $(a, b) = (\sqrt{3}, -1)$  c. $(a, b) = (22\sqrt{3}, 22)$
   d. $(a, b) = (2^{11}\sqrt{3}, -2^{11})$  e. $(a, b) = (2^{10}\sqrt{3}, -2^{10})$

10. Suppose the probability that you will have a good summer is 1 and the probability that you will remember Stokes Theorem in the fall is 0. What is the probability that you will forget Stokes Theorem during a wonderful summer?
    a. -3  b. 1  c. 40  d. 100000  e. Too much information is given to answer this
Free Response: Partial credit given. Show all your work.

1. Find the shortest distance from the origin to the surface $xyz^2 = 2$.

2. Evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy$. 
3. Evaluate the integral \( \int \int_R \sin(9x^2 + 4y^2)\,dA \) where \( R \) is the region in the \( xy \)-plane bounded by \( 9x^2 + 4y^2 = 1 \).

4. Let \( F(x, y) = \frac{-y}{x^2+y^2} \mathbf{i} + \frac{x}{x^2+y^2} \mathbf{j} \) and let \( C \) be the curve \( x^2 + y^2 = 1 \) oriented clockwise. Evaluate the line integral \( \int_C \mathbf{F} \cdot dr \).
5. Consider the vector field $F(x, y, z) = 2xi + 2yj + 2zk$. a) Compute $\text{curl}(F)$.
b) If $C$ is any path from $(0,0,0)$ to $(a_1, a_2, a_3)$ and $a = a_1i + a_2j + a_3k$, prove that $\int_C F \cdot dr = a \cdot a$.

6. Find the simple closed curve $C$ which gives the maximal value of the following integral and explain why it yields the maximal value:

$$\int_C (x^5 - 6y + y^3)dx + (y^4 + 6x - x^3)dy.$$
7. Suppose $F(x, y, z) = (xe^z - 3y)i + (ye^{z^2} + 2x)j + (x^2 y^2 z^2)k$ and $S$ is the portion of the paraboloid $z = 4 - x^2 - y^2$ where $z \geq 0$. Compute $\int_S \text{curl}(F) \cdot dS$.

8. Let $E$ be the region enclosed by the paraboloid $z = 2 - x^2 - y^2$ and the plane $z = 1$. Let $S$ be the surface bounding $E$. Let $F(x, y, z) = (z \tan^{-1}(y^2), z^3 \ln(x^2 + 1), z)$. Find the flux of $F$ across $S$; in other words find $\int_S F \cdot dS$. 
9. Solve $y' + y = \sqrt{x}e^{-x}$ if $y(0) = 3$.

10. Solve the differential equation $y'' + 2y' + y = e^{-x}/x$ subject to the initial conditions $y(1) = 0$ and $y'(1) = e^{-1}$. 