Problem 1. Use Gaussian elimination, find all solutions to the linear system:
\[
\begin{align*}
    x - y + z &= 2, \\
    x + 2y - z &= 3, \\
    2x + y + 3z &= 21.
\end{align*}
\]

Problem 2. Use Taylor approximation to rigorously compute the following limit:
\[
\lim_{x \to 0} \frac{\sin(x) - x - (\log(1 - x))(1 - \cos(x))}{x^3}
\]

Problem 3. Consider the symmetric matrix \( A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \).

1. Write down all eigenvalues of \( A \) and the corresponding eigenvectors. Why does \( A \) have an eigenbasis?
2. Write down a diagonal matrix \( D \) and an invertible matrix \( S \) so that \( A = SDS^{-1} \).
3. Write down a matrix \( B \) so that \( B^8 = A \) (using the first two parts.)

Problem 4. Consider the symmetric matrix \( A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \).

1. Write down the associated quadratic form and show it is positive definite.
2. For what values of \( c \) is the quadratic form \( Q(x, y) = 3x^2 - (5 + c)xy + 2y^2 \) positive definite, positive semidefinite, or indefinite?

Problem 5. Let \( A \) be a symmetric \( n \times n \) matrix with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) and let \( Q \) be the associated quadratic form. Show that, for any vector \( x \in \mathbb{R}^n \),
\[
\lambda_1 \|x\|^2 \leq Q(x) \leq \lambda_n \|x\|^2
\]
where \( \|\| \) is the length of the vector. (Hint: use the spectral theorem to reduce to the diagonal case, and use properties of orthogonal matrices.)

Problem 6. Let \( A \) be a \( n \times m \) matrix with \( n \leq m \). For every choice of \( n \) columns, we have a \( n \times n \) matrix (these are examples of minors of \( A \)). Show that, if there exists some \( n \times n \) minor with a nonzero determinant, then \( A \) has rank \( n \). The other direction is true too, but you don’t have to prove that.

Problem 7. In this question, use Taylor polynomials to prove a ”third-derivative” test for critical points: Suppose we have a \( C^3 \)-function \( f(x) \) such that \( f(0) = 0 \), \( f'(0) = 0 \) and \( f''(0) = 0 \) (so that the second derivative test doesn’t provide any information); suppose also that the third derivative \( f^{(3)}(0) > 0 \).

Show that \( f(x) \) does not have a local maximum or a local minimum at \( x = 0 \) by showing that \( f(x) > 0 \) for \( x > 0 \) sufficiently small, and \( f(x) < 0 \) for \( x < 0 \) and sufficiently small.