Exercise 1. Let \( f : [0,1] \to \mathbb{R} \) be given by \( f(x) = a(x-b)^2 + c \), where \( a, b, c \) are parameters. Find the minimum and maximum of \( f \) depending on the values of \( a, b, c \).

Solution. If \( a = 0 \), the minimum and maximum of \( f \) are \( c \). If \( a \neq 0 \), we have the following cases:

(1) \( a < 0 \). Then

\[
\min(f) = \begin{cases} 
  f(1) = a(b-1)^2 + c & : b \leq 1/2 \\
  f(0) = ab^2 + c & : b > 1/2 
\end{cases}
\]

and

\[
\max(f) = \begin{cases} 
  f(0) = ab^2 + c & : b \leq 0 \\
  f(b) = c & : 0 < b \leq 1 \\
  f(1) = a(b-1)^2 + c & : b > 1 
\end{cases}
\]

(2) \( a > 0 \). Then

\[
\max(f) = \begin{cases} 
  f(1) = a(b-1)^2 + c & : b \leq 1/2 \\
  f(0) = ab^2 + c & : b > 1/2 
\end{cases}
\]

and

\[
\min(f) = \begin{cases} 
  f(0) = ab^2 + c & : b \leq 0 \\
  f(b) = c & : 0 < b \leq 1 \\
  f(1) = a(b-1)^2 + c & : b > 1 
\end{cases}
\]

Exercise 2. (1) Give an example of a function \( f : [0,1] \to \mathbb{R} \) which has a local minimum which is not a global minimum.

(2) Give an example of a function \( f : [0,1] \to \mathbb{R} \) which has an inflection point.

(3) Give an example of a function \( f : (0,1] \to \mathbb{R} \) which does not attain a minimum. What about a function on \([0,1]\)?

Solution. (1) The function

\[ f : [0,1] \to \mathbb{R}, \quad f(x) = x \cos(4\pi x) \]

has a local minimum \( f(1/4) = -1/4 \), but a global minimum \( f(3/4) = -3/4 \).

(2) The function

\[ f : [0,1] \to \mathbb{R}, \quad f(x) = \left( x - \frac{1}{2} \right)^3 \]

has an inflection point at \( 1/2 \).

(3) The functions

\[ f : (0,1] \to \mathbb{R}, \quad f(x) = x \]

\[ g : [0,1] \to \mathbb{R}, \quad g(x) = \begin{cases} 
  1 & : x = 0 \\
  x & : 0 < x \leq 1 
\end{cases} \]

do not attain a minimum. Note that a function on \([0,1]\) that does not attain a minimum is necessarily discontinuous.

Exercise 3. Find the global minimum of the function \( f(x) = x^x \) on \((0,\infty)\). Justify why it is a global minimum.
Solution. Write \( f(x) = e^{x \log(x)} \) and we find
\[
f'(x) = e^{x \log(x)} (\log(x) + 1)
\]
Since \( e^t > 0 \) for all \( t \in \mathbb{R} \), and
\[
\log(x) \begin{cases} 
< -1 & : x < e^{-1} \\ 
= -1 & : x = e^{-1} \\ 
> -1 & : x > e^{-1}
\end{cases}
\]
we find that
\[
f'(x) \begin{cases} 
< 0 & : x < e^{-1} \\ 
= 0 & : x = e^{-1} \\ 
> 0 & : x > e^{-1}
\end{cases}
\]
Hence \( f \) is decreasing on \((0, e^{-1})\) and increasing on \([e^{-1}, \infty)\). It follows that \( f \) has a minimum \( f(e^{-1}) = e^{-1/e} \). This is the unique global minimum since \( f \) is strictly decreasing on \((0, e^{-1})\) and strictly increasing on \((e^{-1}, \infty)\).

Exercise 4. Let \( f : [-3, 3] \to \mathbb{R} \) be given by \( f(x) = x^3/3 + x^2/2 - 2x \). Find the critical points, the local minima/maxima, and the global minima/maxima. What are the global minima/maxima if we take the same function \( f(x) \) on the whole real line?

Solution. (1) Since
\[
f'(x) = x^2 + x - 2 = (x + 2)(x - 1)
\]
the critical points of \( f \) on \((-3, 3)\) are \(-2\) and \(1\). Since
\[
f''(x) = 2x + 1 = \begin{cases} 
-3 & : x = -2 \\ 
3 & : x = 1
\end{cases}
\]
f attains a local minimum \( f(1) = -7/6 \), and a local maximum \( f(-2) = 4/3 \). Note that
\[
f(-3) = \frac{3}{2} \quad \text{and} \quad f(3) = \frac{15}{2}
\]
Thus \( f \) attains global minimum \( f(1) = -7/6 \), and global maximum \( f(3) = 15/2 \).

(2) The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^3/3 + x^2/2 - 2x \) does not have a global minimum or a global maximum. Indeed,
\[
f(x) = \frac{x^3}{3} \left( 1 + \frac{3}{2x} - \frac{3}{2x^2} \right)
\]
For sufficiently large \( x \), \( |3/2x - 3/2x^2| < 1/2 \), so
\[
f(x) \geq \frac{x^3}{3} \left( 1 - \frac{1}{2} \right) = \frac{x^3}{6}
\]
and for sufficiently negative \( x \) (i.e. \(-x\) sufficiently large), again \( |3/2x - 3/2x^2| < 1/2 \), so
\[
f(x) \leq \frac{x^3}{3} \left( 1 - \frac{1}{2} \right) = \frac{x^3}{6}
\]
Thus \( f \) does not attain a global minimum or a global maximum.

Exercise 5. Let \( f : [0, \infty) \to \mathbb{R} \) be a continuous function. Let \( y \in [0, \infty) \); we can define a new function \( g(y) \) by setting \( g(y) \) to be the global minimum of the function \( f \) on the interval \([0, y]\). Why does this minimum exist? Show that \( g : [0, \infty) \to \mathbb{R} \) is a decreasing function.

Solution. By definition,
\[
g(y) = \min_{x \in [0, y]} f(x)
\]
which exists because \( f \) is a continuous function and \([0, y]\) is a closed, bounded interval. Given \( y_1 \geq y_0 \), we have \([0, y_0] \subset [0, y_1]\). Suppose \( \min_{x \in [0, y_0]} f(x) = f(x_0) \) for some \( x_0 \in [0, y_0] \). Then \( x_0 \in [0, y_1] \), and
\[
\min_{x \in [0, y_1]} f(x) \leq f(x_0) = \min_{x \in [0, y_0]} f(x)
\]
This shows that \( g(y_1) \leq g(y_0) \), so \( g \) is a decreasing function.
**Exercise 6.** Find the rank of the matrix

\[
A = \begin{pmatrix}
1 & 0 & -1 & 2 \\
-2 & 1 & 1 & -1 \\
-3 & 2 & 1 & 0
\end{pmatrix}
\]

**Solution.** Since the rank of \(A\) is equal to the maximal number of linearly independent rows, it is invariant under elementary row operations. Thus,

\[
\text{rank}(A) = \text{rank}
\begin{pmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & -2 & 6
\end{pmatrix} = \text{rank}
\begin{pmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & -1 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix} = 2
\]

where in the first equality, we added 2 and 3 copies of the first row to the second and third rows, respectively, and in the second equality, we subtracted 2 copies of the second row from the third row.

**Exercise 7.** Let \(A\) be an invertible \(n \times n\) matrix and \(v_1, \ldots, v_k \in \mathbb{R}^n\) be \(k\) linearly independent vectors. Show that \(Av_1, Av_2, \ldots, Av_k \in \mathbb{R}^n\) are also linearly independent.

**Solution.** Suppose \(c_1, \ldots, c_k \in \mathbb{R}\) satisfy the equation

\[
c_1 Av_1 + c_2 Av_2 + \cdots + c_k Av_k = 0
\]

We have to show that \(c_1 = c_2 = \cdots = c_k = 0\). Indeed, since \(A\) is linear, the above equation implies

\[
A(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) = 0
\]

Since \(A\) is invertible, we can multiply both sides of the equation by \(A^{-1}\), so

\[
c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0
\]

By the hypothesis that \(v_1, v_2, \ldots, v_k\) are linearly independent, we obtain \(c_1 = c_2 = \cdots = c_k = 0\). \(\Box\)