ANALYSIS AND OPTIMIZATION, SOME PRACTICE
PROBLEMS

Problem 1. Write down a function on \( \mathbb{R}^2 \) with a critical point at \((0,0)\) that is neither a local minimum or local maximum. Write down a function whose gradient at \((0,0)\) is \((1,3)\) and whose Hessian is \[
\begin{bmatrix}
2 & 1 \\
1 & 8
\end{bmatrix}
\]

Problem 2. Find the global minimum and maximum of the function \( f : [-4,4] \to \mathbb{R} \) given by \( f(x) = x^4 - 4x^3 + 4x^2 + 6 \).

Problem 3. Find all critical points of the function \( f(x,y,z) = x^4 + y^4 - 4xy - 2z^2 + 1 \). For each one, write down the Hessian at that point and use the second derivative test to see if it is a local minimum, local maximum, or saddle point (or say if the test is not enough!)

Problem 4. Find the rank of the matrix \( A = \begin{bmatrix}
1 & -2 & 1 & 0 \\
3 & -6 & 2 & -1 \\
-2 & 4 & 0 & 2
\end{bmatrix} \). Use Gaussian elimination to solve the equation \( A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \)

Problem 5. State the spectral theorem. What is the meaning of an orthogonal matrix? What property do the eigenvectors of a symmetric matrix satisfy? Write down eigenvectors and eigenvalues for the symmetric matrix:
\[
A = \begin{bmatrix}
0 & 2 & 2 \\
2 & 1 & 0 \\
2 & 0 & -1
\end{bmatrix}
\]
Write down the quadratic form associated to the symmetric matrix \( A \).

Problem 6. Using the principal minor criterion, test if the matrix \( A = \begin{bmatrix}
2 & 1 & 0 \\
1 & 4 & 1 \\
0 & 1 & 3
\end{bmatrix} \)
is positive definite, negative definite, or indefinite (or none of the above).

Problem 7. State Taylor’s theorem for second order approximation for a one-variable function \( f(x) \) (i.e. up to the second derivative). Calculate the third order Taylor approximation for \( f(x) = e^{2x-1} \cdot \sin(x) \).

Problem 8. State the definition of a convex set. Let \( S \subset \mathbb{R}^n \) be a convex set. Given \( x, y, z \in S \) and three positive numbers such that \( a + b + c = 1 \), show that \( ax + by + cz \) is in \( S \) also.

Problem 9. Show the following: If \( A \) is a \( n \times n \) symmetric matrix which is negative semi-definite, then either it is negative definite or its rank is less than \( n \).
Problem 10. Let $A$ be a $3 \times 3$ symmetric matrix which is not a diagonal matrix. Show that its eigenvalues are not all the same. Let $Q(x)$ be the corresponding quadratic form: Show that 

$$\lim_{x \to 0} \frac{Q(x)}{\|x\|^2}$$

does not exist!