1. Differentiate the identity $xy = 8$ with respect to $t$ to obtain

$$x \frac{dy}{dt} + y \frac{dx}{dt} = 0$$

The problem statement says that $x = 4$, $y = 2$, and $\frac{dy}{dt} = -3$, and asks to find $\frac{dx}{dt}$. Substituting these known values and solving for $\frac{dx}{dt}$ in the equation above gives

$$\frac{dx}{dt} = -\frac{x}{y} \frac{dy}{dt} = -\frac{4}{2} \cdot (-3) = 6$$

2. 
   a) $\frac{1}{2}$
   b) $\frac{1}{2}$ (The derivative of $\arctan(x)$ is $\frac{1}{1+x^2}$)
   c) $(f(g(t)))’ = f’(g(t))g’(t)$ and $\frac{dy}{dt} = \frac{df}{dg} \frac{dg}{dt}$
   d) If we set $g(t) = e^t$, and $f = \arctan$, then a), b), and c) together tell us that $\frac{d}{dt} \arctan(e^t)$ at $t = 0$ is $1 \cdot \frac{1}{2}$. We start at $\arctan(e^0) = \frac{\pi}{4}$, and increase by an amount 0.1 at a rate of $\frac{1}{2}$, so we end up at $\frac{\pi}{4} + 0.05$.

3. 
   a) The tangent line of the function $f$ at the point $a$ is $f(a) + f’(a)(x - a)$. In this case, $f(a) = \log(\alpha \cdot 0 + 1) = 0$ and $f’(a) = \frac{\alpha}{\alpha \cdot 0 + 1} = \alpha$, so the linear approximation is $y = \alpha x$.
   b) Write $f(x) = \alpha x + \text{error}$. The limit is

$$\lim_{x \to 0} \frac{\alpha x + \text{error}}{x} = \alpha + \lim_{x \to 0} \frac{\text{error}}{x}$$

If the second limit is 0. The only way it is something other than 0 is if error looks something like $\beta x$, but then that would be part of the linear approximation (since it’s a line). Hence error = $o(x)$, and the limit on the right is 0.

4. 
   a) The rate of change of $\sqrt{x}$ at $x = 9$ is $\frac{1}{2} \cdot \frac{1}{\sqrt{9}} = \frac{1}{6}$. So $\sqrt{9.01}$ is going to be $3 + 0.01 = 3.00166666...$. The actual answer is 3.00166662....
   b) We should still approximate by the tangent line at $x = 9$, since that makes numbers easy. The rate of change is still $\frac{1}{6}$, so our estimate is 3.1666... This is too high, since $\sqrt{x}$ is concave down. I would guess that it’s too high by 0.01. The actual answer is 3.1622..., so the estimate is a fair bit better than I thought it was!
5.

a) There are several ways to do this. One way is to square both sides to get

\[ 2x^6 + 3x^2 + \ldots = 2x^6 + (O(x))^2 + x^3O(x) = 2x^6 + O(x^2) + O(x^4) = 2x^6 + O(x^4) \]

Since \(3x^2 = O(x^4)\) and every step is reversible, this proves the statement.

Another way to do it is to factor out the leading term from the square root to get \(\sqrt{2x^3}\sqrt{1 + \frac{3}{2x^2}}\). Linear approximation says that \(\sqrt{1+u} = 1 + O(u)\), so in this case we get \(\sqrt{2x^3} + x^3O\left(\frac{1}{x^2}\right) = \sqrt{2x^3} + O\left(\frac{1}{x}\right)\), which is substantially better than what we need.

You could also use the definition of big-O notation, by showing

\[ \lim_{x \to \infty} \frac{\sqrt{2x^6 + 3x^4 - 4} - \sqrt{2x^3}}{x} < \infty \]

and this isn’t too hard, but I prefer the other methods, personally.

The analogous statement would be \((3x^9 + 4x^8 - x^2 - 100)^{\frac{1}{5}} = 3^{\frac{1}{5}}x^\frac{9}{5} + O(x^\frac{8}{5})\) (in both cases the approximation we’re making is that the square root/fifth root is linear).

b) \[ \lim_{x \to \infty} \frac{\sqrt[5]{2x^3} + O(x)}{x^3} = \sqrt[5]{2} + \lim_{x \to \infty} \frac{O(x)}{x^3} = \sqrt[5]{2} + 0 \]

c) \[ \lim_{x \to \infty} \frac{3^{\frac{1}{5}}x^\frac{9}{5} + O(x^{\frac{8}{5}})}{x^r} = \lim_{x \to \infty} 3^{\frac{1}{5}}x^{\frac{9}{5}-r} + O(x^{\frac{8}{5}-r}) \]

If \(r > \frac{9}{5}\), then limit goes to 0. If \(r < \frac{9}{5}\), then the limit goes to infinity. If \(r = \frac{9}{5}\), then the limit is \(3^{\frac{1}{5}}\). The secondary term doesn’t end up mattering.