Section 1.1

Exercise (42). *Find a system of linear equations with three unknowns whose solutions are the points on the line through (1,1,1) and (3,5,0).*

Solution. The line through (1,1,1) and (3,5,0) can be parametrized by the following three equations:

\[
\begin{align*}
  x &= 2t + 1 \\
  y &= 4t + 1 \\
  z &= -t + 1
\end{align*}
\]

Solving the last equation for \( t \) gives \( t = -z + 1 \). We can then plug this in for \( t \) in the first two equations to get

\[
\begin{align*}
  x &= 2(-z + 1) + 1 \\
  y &= 4(-z + 1) + 1
\end{align*}
\]

Finally, we put this linear system in standard form

\[
\begin{align*}
  x + 2z &= 3 \\
  y + 4z &= 5
\end{align*}
\]

We check this solution by first noting the both of the given points satisfy these equations. Then we check that the two equations describe two different planes in \( \mathbb{R}^3 \). This is is shown by finding a point that satisfies one of the
equations, but not the other; (1, 0, 1) is such a point. Thus we have two distinct plains intersecting in $\mathbb{R}^3$ so we know their intersection is a line. Because (1,1,1) and (3,5,0) are on both planes, the intersection line must be the line through the given points.

Section 1.2

Exercise (48). For an arbitrary positive integer $n \geq 3$, find all solutions $x_1, x_2, \ldots, x_n$ of the simultaneous equations

\[
\begin{align*}
  x_2 &= \frac{1}{2}(x_1 + x_3) \\
  x_3 &= \frac{1}{2}(x_2 + x_4) \\
  &\vdots \\
  x_{n-1} &= \frac{1}{2}(x_{n-2} + x_n)
\end{align*}
\]

Solution. We begin by considering the case when $n = 3$. This will not give a complete solution, but will provide insight into the general case. When $n = 3$, there is only one equation in the linear system

\[
x_2 = \frac{1}{2}(x_1 + x_3).
\]

In standard form, this equations becomes

\[
\frac{1}{2}x_1 - x_2 + \frac{1}{2}x_3 = 0
\]

and solving for $x_1$ gives

\[
x_1 = 2x_2 - x_3.
\]

The parametrized solution is then

\[
\begin{align*}
  x_1 &= 2t - s \\
  x_2 &= t \\
  x_3 &= s
\end{align*}
\]

The solution has one leading variable and two free variables.

Before solving the system for a general $n$, we will consider one more special case: $n = 4$. In this case the linear system consists of the two equations

\[
x_2 = \frac{1}{2}(x_1 + x_3)\quad\text{and}\quad x_3 = \frac{1}{2}(x_2 + x_4)
\]
In standard form, these equations become
\[ \frac{1}{2}x_1 - x_2 + \frac{1}{2}x_3 = 0 \text{ and } \frac{1}{2}x_2 - x_3 + \frac{1}{2}x_4 = 0. \]

From these equations, we can determine the augmented matrix of the linear system
\[ \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \end{bmatrix} \]

To row reduce this matrix, multiply both rows by 2 and then add twice the second row to the first row. The reduce row echelon form is then
\[ \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{bmatrix} \]

This gives the following parametrized solution:
\[ \begin{align*} x_1 &= 3t + 2s \\ x_2 &= 2t + s \\ x_3 &= t \\ x_4 &= s \end{align*} \]

Now we will consider the system for a general n. The basic strategy will be the same: find the augmented matrix of the system, row reduce it and then find the parametrized solution. First, we put the linear equations in standard form.
\[ \begin{align*} \frac{1}{2}x_1 - x_2 + \frac{1}{2}x_3 &= 0 \\ \frac{1}{2}x_2 - x_3 + \frac{1}{2}x_4 &= 0 \\ &\vdots \\ \frac{1}{2}x_{n-2} - x_{n-1} + \frac{1}{2}x_n &= 0 \end{align*} \]

The augmented matrix of this system is then
\[ \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & \frac{1}{2} & 0 \end{bmatrix} \]
We begin row reducing this matrix by multiplying all the rows by 2. There is a leading one in each row and in the first \( n - 2 \) columns.

\[
\begin{bmatrix}
1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\
\end{bmatrix}
\]

We begin to clear the columns with leading ones by adding twice the second row to the first row. At this point the first two columns are done; they each contain a single leading one and no other nonzero entries.

\[
\begin{bmatrix}
1 & 0 & -3 & 2 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\
\end{bmatrix}
\]

The third column can be cleared by adding the thrice the third row to the first row and twice the third row to the second row.

\[
\begin{bmatrix}
1 & 0 & 0 & -4 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -3 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\
\end{bmatrix}
\]

Continuing to apply row operations in this manner will put the matrix in reduced row echelon form. The complete sequence of row operations can be summarized as follows:

row 1 = \( 2(r_1 + 2r_2 + 3r_4 + \cdots + (n - 2)r_{n-2}) \)

row 2 = \( 2(r_2 + 2r_3 + 3r_4 + \cdots + (n - 3)r_{n-2}) \)

\vdots

row \( n - 3 \) = \( 2(r_{n-3} + 2r_{n-2}) \)

row \( n - 2 \) = \( 2(r_{n-2}) \)

Here \( r_i \) indicates the \( i^{th} \) row. We can check that these row operations have the desired affect on the middle columns by considering their affect on three consecutive rows.

\[
\begin{bmatrix}
\cdots & m & -m + 1 & 0 & -m - 3 & m + 2 & \cdots \\
\end{bmatrix}
\]
The resulting row reduced matrix is then.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 1-n & n-2 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & -n & n-1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0
\end{bmatrix}
\] \quad (2)

The following parametrized solution describes all possible solutions to the general linear system.

\[
\begin{align*}
x_1 &= (n-1)t + (2-n)s \\
x_2 &= nt + (1-n)s \\
\vdots \\
x_{n-3} &= 3t - 2s \\
x_{n-2} &= 2t - s \\
x_{n-1} &= t \\
x_n &= s
\end{align*}
\]

**Exercise (56).** Find the conics through the points (0,0), (1,0), (0, 1) and (1, -1). Draw a rough sketch of your solutions curve(s).

**Solution.** We are given that a conic is described by an equation of the form

\[
c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 x y + c_6 y^2 = 0
\]

Plugging each of the points into this equation gives the linear system

\[
\begin{align*}
c_1 &= 0 \\
c_1 + c_2 + c_4 &= 0 \\
c_1 + c_3 + c_6 &= 0 \\
c_1 + c_2 - c_3 + c_4 - c_5 + c_6 &= 0
\end{align*}
\]

The augmented matrix of this linear system is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -1 & 1 & -1 & 1 & 0
\end{bmatrix}
\]
We begin to row reduce this matrix by subtracting the first row from each of the rows below it.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & -1 & 1 & -1 & 1 \\
\end{bmatrix}
\]

Next, we subtract the second from the forth row and add the third row to the forth row.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

Finally, we multiply the fourth row by \(-1\) to achieve reduced row echelon form.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 \\
\end{bmatrix}
\]

The parameterized solution of the linear system is

\[
\begin{align*}
c_1 &= 0 \\
c_2 &= -t \\
c_3 &= -s \\
c_4 &= t \\
c_5 &= 2s \\
c_6 &= s
\end{align*}
\]

Thus the conics that pass through the given points are described by equations of the form

\[-tx - sy + tx^2 + 2sxy + sy^2 = 0\]

Figure 1 is a graph of a few of these conics.
Figure 1: Conics through (0,0), (1,0), (0, 1) and (1, -1)