Note Stanley's convention

\[ \hat{P} \text{ poset, } \hat{\hat{P}} \text{ has } \hat{\hat{0}}, \hat{\hat{1}} \text{ adjoined (needed or not?)} \]

so \[ \tilde{\chi}(\hat{\hat{P}}) = \tilde{\chi}(\Delta(\hat{\hat{P}})) \]

this is same as us, because \((\hat{\hat{0}}, \hat{\hat{1}}) \text{ is } P\)

**Geometric realization** \(X = |\Delta| \text{ of } \Delta\)

\[ \tilde{\chi}(X) = \sum_i (-1)^i \tilde{h}_i(X, \mathbb{Z}) \]

(compute any way you like singular homology of \(X\)

simplicial \(\Delta)\)

so \(\mu^\hat{\hat{P}}(\hat{\hat{0}}, \hat{\hat{1}}) \text{ depends only on } |X| \text{ not } X\)

(\"homotopy equivalence of posets" sneaking in\"")

---

**Finite regular cell complex** \(\Gamma = \text{union nonempty pairwise disjoint}\)

open cells \(\sigma_i \subset \mathbb{R}^n\) so

(a) \((\overline{\sigma_i}, \overline{\sigma_i} - \sigma_i) \cong (B^n, S^{n-1}) \text{ some } n = n(i)\)

(b) each \(\overline{\sigma_i} - \sigma_i\) is union of \(\sigma_j\)'s

**Cellular homology generalizes simplicial homology**

need sign for \(\sigma_j \subset \sigma_i\) facet, sign not obvious

any two "good" sign rules equivalent

(see Bruns, Herzog e.g.)
\[ |\Pi| \text{ realization of } \Pi \]

\[ \text{sd}(\Pi) \] simplicial complex of chains of cells \( \sigma_1 \subset \sigma_2 \subset \ldots \sigma_k \) of \( \Pi \)

first barycentric subdivision

\[ \mathcal{P}(\Pi) \] facet poset of cells of \( \Pi \)

(don't include empty face \( \emptyset \))

Then

\[ \text{sd}(\Pi) = \Delta(\mathcal{P}(\Pi)) \]

so

\[ M^{\hat{\rho}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Pi) \]

\[ \begin{align*}
- \text{empty} &= -1 \\
+ V &= 5 \\
- E &= -6 \\
+ F &= 1 \\
\end{align*} \]

\[ \tilde{\chi} = -1 \]

Example

\[ \begin{align*}
\text{sd}(\Pi) &= \Delta(\mathcal{P}(\Pi)) \\
\text{h}_k(s') &= 0 \text{ for } k \\
\text{h}_1(s') &= 1
\end{align*} \]

\[ \mathcal{P}(\Pi) \]

\[ \begin{array}{cccc}
\hat{1} & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 \\
\end{array} \]

maximal chains length 1

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & \\
1 & 2 & 3 & 4 & 5 \\
\end{array} \]

recursively compute \( M^{\hat{\rho}}(\hat{0}, \hat{1}) \)

\[ M_{xy} = - \sum_{x \leq z \leq y} M_{xz} \]

\[ \Rightarrow M(\hat{0}, \hat{1}) = -1 \]

\[ \mu(\hat{0}, x) \text{ for } x \in \hat{P} \]
what about other $M_\delta(x,y)$?

link $lk F$ of a face $F \in \Delta$, a simplicial complex, defined by

\[ lk F = \{ G \setminus F | F \subseteq G \in \Delta \} \quad \text{(restated slightly from Stanley)} \]

\[ \text{in closed manifold, } \quad \dim F = i \quad \dim \Delta = n \quad \Rightarrow \quad \dim lk F \leq n-i-1 \]

\[ part \ of \ \Delta, \ \dim 3 \quad \quad \quad \quad \quad \quad F, \ \dim 1 \quad \quad \quad \quad \quad \quad lk F \cong S^{1} \ (1\text{-sphere}) \]

Simplicial manifold, PL category, becomes our "definition" of a manifold (WARNING: higher dims, wild triangulations of manifolds this fails) but true at level of homology.

\[ \text{part of } \Delta, \ \dim 3 \quad \quad \quad \quad \quad F, \ \dim 2 \quad \quad \quad \quad \quad lk F \cong S^{0} \ (2\text{ points}) \]

\[ \Pi \text{ finite regular cell complex, } |\Pi| = \text{manifold} \]

\[ M_\delta(x,y) = \begin{cases} \emptyset, & x \neq \hat{\delta}, y = \hat{1}, \ x \text{ lies on boundary of } |\Pi| \ \checkmark \\ x(\Pi), & x = \hat{\delta}, y = \hat{1} \ \checkmark \\ (-1)^{\ell(x,y)} \ \text{otherwise} \end{cases} \]
we use $1kF$ to express $\mu_\mathcal{G}(x_1y_1)$:

given $x < y$ choose saturated chains in $P$ (not $\hat{P}$)

\[
\begin{align*}
& x_1 < x_2 < \ldots < x_r = x < \\
& \quad \min y_1 < \ldots < y_s = \max
\end{align*}
\]

then $\Delta((x_1y_1)) = 1kF$ in $\Delta(P)$

over our notation so far

$F = \xi x_1, \ldots, x_r, y_1, \ldots, y_s$

( so the family of spaces $\Delta((x_1y_1))$ all related, as links of $\Delta(P)$)

Now, $x$ lies on boundary of $\Gamma$

\[
\begin{align*}
\text{Choose } & x_1, x_2, x_3 \\
& x_1 < x_2 < x_3 = x \\
& \text{still boundary cell} \\
\implies & 1kF \approx B^i \text{ some } i
\end{align*}
\]

$\implies \mu_\mathcal{G}(x_1y_1) = \tilde{x}(\Delta((x_1y_1)))$

$= \tilde{x}(1kF) = 0 \quad \Box$
remaining case: \( \mu_P(x,y) = (-1)^{\ell(x,y)} \)

\( x \neq \hat{0} \quad y \neq \hat{1} \quad \text{or} \quad x \text{ not on boundary} \)

How to think of this: \( \hat{1} \) has "boundary" \( \Gamma \)

any other \( y \) has boundary \( \approx \delta_i \) some \( i \)

\( L_K F \) is usual link of \( x \) in boundary of \( y \)

so \( \chi(L_K F) \) is \( \pm 1 \), depends on \( \dim \) of link

homology of a sphere

\[ \implies \text{Def } P \quad \text{[with } \hat{0}, \hat{1} \text{]} \quad \text{semi-Eulerian} \]

if \( \mu_P(x,y) = (-1)^{\ell(x,y)} \) for \( (x,y) \neq (\hat{0}, \hat{1}) \)

Eulerian

if also \( \mu_P(\hat{0}, \hat{1}) = (-1)^{\ell(\hat{0}, \hat{1})} \)

\[ |\Gamma| \text{ manifold w/ boundary } \implies \hat{P}(\Gamma) \text{ semi-Eulerian} \]

\[ |\Gamma| \text{ sphere } \implies \text{ Eulerian} \]

Example \( B_n \) Eulerian

\[ B_n \cong 2^n \quad s_i T \cap n \]

\[ \mu(T_i, s) = (-1)^{1-s_i-1} \]

Recall

Indeed \( B_n = \hat{P}(\Gamma) \), \( \Gamma \) boundary complex of \( (n-1)\)-simplex

(subsets of \( n \quad \iff \quad \text{faces of simplex} \)

\( \dim \))
aside: upper bound conjecture for spheres

Given any simplicial sphere, d vertices $S^{n-1}$, pick d generic points on rational normal curve
\[ \gamma(t) = (t, t^2, \ldots, t^n) \] $\to \mathbb{R}^n$
and take $C_{d,n} =$ boundary of convex hull of
\[ \frac{1}{z} \gamma(t_1), \ldots, \gamma(t_d) \]
then face numbers bounded by those of $C_{d,n}$

(external case) (messy but worked out) (beautiful in eyes of beholder)

proved P. McMullen for boundary of polytopes
R. Stanley for abstract simplicial spheres

Famous proof, tied together commutative algebra w/ combinatorics
"taught combinatorists (and others) the phrase"

Cohen-Macaulay

in algebraic geometry, many implies

simplest (?) \( I \subset S = \mathbb{K}[x_0, \ldots, x_n] \) defines $X \subset \mathbb{P}^n$

$S/I$ Cohen-Macaulay $\iff$ chain of syzygies no longer
same length as a complete intersection same codim as $X$

$\iff X \cap L$ pure dim (no embedded components)
for generic linear spaces $L \subset \mathbb{P}^n$
moving toward combinatorics, cell complexes:

- take polyhedral cone, \( \mathbb{T} \)
- \( \mathbb{T}^\circ \) = semigroup of lattice points
- \( \mathbb{S}^0 \) (holes in lattice \( \rightarrow \) not Cohen-Macaulay)
- \( k[T] = S/I \) semigroup algebra, toric ring
- Cohen-Macaulay defines \( X \) toric variety (needs choice of gens of \( T \))
- \( \Gamma = \) boundary of slice of cone \( \cong S^i \) some \( i \)

\[ \Rightarrow H^\ast(\Gamma) = \begin{cases} K, & * = i \\ 0, & \text{else} \end{cases} \] computes sheaf cohomology of \( X \),

0 except \( \dim X \)

So seeing (co)homology ranks of 0, except \( \ast \) in top \( \dim \)

\[ \Rightarrow \text{Pavlovian response, "there must be something Cohen-Macaulay going on here"} \]

response is usually justified, and rewarded.

**Def** finite poset \( P \) Cohen-Macaulay (over \( \mathbb{Q} \))

if for all \( x < y \) in \( P \),

look at \( \Delta((x,y)) \), (order complex of open intervals)

\( \tilde{H}_i(\Delta((x,y)), \mathbb{Q}) = 0, \ i < \dim \Delta((x,y)) \)

\( \frac{l(x,y)}{\dim 0, l(x,y) = 2} \)

\( x < z < y \)
\[ \text{def} \quad M_\rho \text{ alternates in sign } \iff (-1)^{g(x,y)} M_\rho(x,y) \geq 0 \]

\[ P \text{ CM } \Rightarrow \quad M_\rho \text{ alternates in sign} \]
\[ M_\rho(x,y) = \tilde{\Sigma}(\mathcal{L}(x,y)) = (-1)^{\dim L_\rho}(\mathcal{L}(x,y), G) \]

so

examples of CM posets \{ P(\Pi), \ "nice" manifolds with boundary finite semimodular lattices \}

we never looked at these yet.

3.9 Lattices and their Möbius Algebras

special methods for lattices

Möbius algebra of L lattice / K field \( A(L, K) \)

= \( K[L] \)

view meet operation as sum for \( L \)

semi-group algebra

familiar example \([\mathbb{N}^n, \text{ lattice points in first orthant } ] \)

separate \([\mathbb{N}^n, +] \) usual semigroup \( (\mathbb{N}^n, \times \Lambda) = (\mathbb{N}^n, \gcd) \) Möbius algebra

meet \( \leq \) intersection
Thue, Feboz

so $A(N^n, k)$ looks like polynomial ring, except multiplication is gcd operation. (!)

not a quotient of a finite dim poly ring

idempotent basis $x \in L$ for commutative $A(L, k)$

want to make $A(L, k) \cong k^L$ more explicit:

$L$ finite

for $x \in L$ define

$$\delta_x \in A(L, k) \text{ by } \delta_x = \sum_{y \leq x} \delta(y) \mu(y, x)y$$

$$\Rightarrow \quad \boxed{x = \sum_{y \leq x} \delta_y} \quad \text{(very cool) (**)}

(**) span, right number $\Rightarrow$ $\delta_x$'s form basis for $A(L, k)$

check (**):

$$\sum_{y \leq x} \delta_y = \sum_{y \leq x} \sum_{z \leq y} \mu(z, y)z$$

$$= \sum_{z \leq x} \left( \sum_{z \leq y \leq x} \mu(z, y) \right) z = x$$

$$0, \ z < x$$

$$1, \ z = x$$

Theorem $L$ finite lattice,

$A'(L, k) = \bigcup_{x \in L} K_x, \ K_x \cong k$

$\delta'_x$ identity elem of $K_x$ \hspace{1cm} $\delta'_x \delta'_y = \begin{cases} 0, \ x \neq y \\ \delta'_x, \ x = y \end{cases}$

Then $\Theta: A(L, k) \to A'(L, k)$ algebra isomorphism
proof

look at map  \( x \rightarrow x' \)

\[
\prod_{y \leq x} \delta_y = \prod_{y \leq x} \delta_y'
\]

... need only show \( x'y' = (x \land y)' \)

\[
x'y' = (\sum_{z \leq x} \delta_z')(\sum_{w \leq y} \delta_w') = \sum_{z \leq x} \delta_z' = \sum_{z \leq x \land y} \delta_z' = (x \land y)'
\]

Corollary

Let finite lattice, \( \hat{1} \neq a \)

\[
\sum_{x \mid x \land a = \hat{0}} \mu(x, \hat{1}) = 0
\]

\((++)\)

\(a = \hat{0}\) is usual \( \sum_{x} \mu(x, \hat{1}) = \hat{0} \), not new. \( a \neq \hat{0}\) new, fewer terms

proof in \( A(L, C) \),

\[
\begin{align*}
\alpha \delta_{\hat{1}}' &= (\sum_{b \leq a} \delta_{b}) \delta_{\hat{1}} = 0 \quad \text{since } a \neq \hat{1} \\
\alpha \delta_{\hat{1}} &= \alpha \sum_{x \in L} \mu(x, \hat{1}) x = \sum_{x \in L} \mu(x, \hat{1}) a \land x
\end{align*}
\]

The \( \hat{0} \) coefficient (in expansion \( \sum_{x \in L} c_x x \))

is \((++)\) as above, claimed.