3.5 Chains in distributive lattices

\[ \# \text{k-elem order ideals } \mathcal{I} \subseteq P \]
\[ \Rightarrow \# \text{elems of } \mathcal{J}(P) \text{ of rank } k \quad \text{is defined!} \]

\[ \# \text{k-elem anti-chains of } P \]
\[ = \# \{ x \in \mathcal{J}(P) \mid x \text{ covers exactly } k \text{ elems} \} \]

3.5.1 P finite poset, m \geq 0

(a) \( \# \{ \sigma : P \to m \text{ order preserving} \} \)
\[ = (b) \# \{ \emptyset \subseteq I_0 \subseteq I_1 \subseteq \ldots \subseteq I_m = \hat{1} \} \text{ in } \mathcal{J}(P) \]
\[ = (c) \# \mathcal{J}(P \times m-1) \]

Order ideal in \( P \times m-1 \) is a nested chain of order ideals in \( P \):

\[ P = \begin{array}{c}
\begin{array}{c}
\hat{0} = I_0 \subseteq I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 = \hat{1}
\end{array}
\end{array} \]

\( P \times 3 \)

So \( I \subseteq P \times m-1 \) given by \( I = \bigcup_{i=1}^{m-1} \text{Im}_i \times i \)
3.5.2

(a) \# surjective order-pres \sigma : P \to m

= (b) \# chains \hat{0} = I_0 < I_1 < \cdots < I_m = \hat{1} \text{ in } \mathcal{J}(P)

\text{if } \#P = n, \quad e(P) = \# \sigma : P \to n \wedge \text{order pres bijection}

- extensions of } P \text{ to total order
- linearizations of } P
- linear extensions of } P

\text{e}(P) = \# \text{ maximal chains in } \mathcal{J}(P)

= \# \text{ facets in } \Delta(\mathcal{J}(P)) \quad \text{(pure dim simplicial complex)}

= \text{degree of corresponding variety of face ring of } \Delta(\mathcal{J}(P)) \quad \text{chain ring of } \mathcal{J}(P)

lattice paths

linear extension is restricted permutation

\# \Pi \in S_5 \text{ so}

\begin{align*}
\Pi(1) &< \Pi(3) \\
\Pi(2) &< \Pi(3) \\
\Pi(2) &< \Pi(4) \\
\Pi(4) &< \Pi(5)
\end{align*}

\begin{align*}
\Pi & = 390 \quad 105 \\
\text{= e}(P)
\end{align*}
Dilworth Theorem

\[
\min k : P = C_1 \cup \ldots \cup C_k \text{ partition into chains} = \max k : k = \# \text{ antichain of } P
\]

reminds us of "min cut/max flow" theorems for network flows

![Directed graph with source, sink, directed edges, and capacities](image)

Each edge has capacity, max flow = source to sink = min \# cut

oriented matroid theory, linear programming duality

given \( P = C_1 \cup \ldots \cup C_k \) (not nec. minimal)

define \( S : I(P) \to \mathbb{N}^k \) by

\[
S(I) = (\#\text{Inc}_{C_1}, \#\text{Inc}_{C_2}, \ldots, \#\text{Inc}_{C_k})
\]

\[
S(\emptyset) = \langle 0 \rangle S(\hat{I}) = (\#C_1, \ldots, \#C_k)
\]

\[
I \subseteq J \in J(P) \implies J = I \cup \exists x^3, x \in C_i \\
J \text{ covers } I \\
\implies S(J) - S(I) = (0, \ldots, 0, 1, 0, \ldots, 0)
\]

so \( S \) is cover-preserving \( \Rightarrow \) rank-preserving, injective lattice homomorphism
Define $\Gamma_0 \subset \mathbb{R}^k$ by

\begin{align*}
&\text{(for each } [I,J] \text{ interval in } J(P) \\
&\text{take convex hull of image in } \mathbb{N}^k) \tag{2} \\
&\text{union is compact polyhedral set,} \\
&\text{# lattice paths from } \vec{0} \text{ to } \vec{s}(I) \text{ in } \Gamma_0 \\
&= \# \text{ maximal chains in } J(P) \\
&= e(P)
\end{align*}

**Example:**

\[
\# \{ \Pi \in \mathcal{B}_5 \mid \begin{array}{lll}
\Pi(1) < \Pi(3) \\
\Pi(2) < \Pi(3) \\
\Pi(2) < \Pi(4) \\
\Pi(4) < \Pi(5)
\end{array} \}
\]

\[
= e(P), \quad P = \begin{array}{c}
95 \\
36 \\
10 \\
02
\end{array}
\]

\[
= \# \text{ maximal chains in } J(P)
\]

Write $P = C_1 \cup C_2$, $\begin{array}{c}
95 \\
36 \\
10 \\
02
\end{array} = \begin{array}{c}
95 \\
34 \\
10 \\
02
\end{array}$

\[
J(P) \hookrightarrow \mathbb{N}^2 \text{ and } \Gamma_0
\]
we have this technology

\[
\begin{vmatrix}
#1 & #1_{12} & #1_{42} \\
#2_{10} & #2_{101} & #2_{102}
\end{vmatrix} = \begin{vmatrix}
(3) & 1 \\
1 & 1
\end{vmatrix} = 10 - 1 = 9
\]

because permuting ends forces overlaps, whenever points to avoid.

(could do inclusion-exclusion by "almost" triangular matrix)
ex: \( P = c_1 + c_2 \) (disjoint union)

\[
\begin{array}{c}
\text{m} \\
\text{n} \\
\text{p}
\end{array}
\Rightarrow
\begin{array}{c}
\text{oo} \\
\text{mn}
\end{array}
\]

\( J(p) \quad e(p) = \binom{m+n}{m} \)

\[
p = p_1 + \ldots + p_k \quad \#p_i = n_i
\]

\[\Rightarrow e(p) = \left( \binom{n_1 + \ldots + n_k}{\#n_1 \ldots n_k} \right) e(p_1) \ldots e(p_k)\]

reminiscent of multiset permutations

\[
M = \{1^{n_1}, \ldots, k^{n_k}\} \quad n = \sum n_i
\]

\[\Rightarrow \emptyset: \mathcal{S}(m) \times \mathcal{S}_{n_1} \times \ldots \times \mathcal{S}_{n_k} \rightarrow \mathcal{S}_n\]

is a bijection

so above formula is restricted version of same construction.
Ex: \( P = 2 \times n \)  
\[ C_1 = \sum \binom{2}{2} + \binom{2}{1} \]
\[ C_2 = \sum \binom{2}{2} + \binom{2}{1} \]

Why did Stanley switch 1, 2 here and drawing?

\( I \subseteq J(P) \)

Can't have \( \# I \cap C_2 > \# I \cap C_1 \)

\( I \subseteq J(P) \), can't have \( \# I \cap C_2 > \# I \cap C_1 \), but otherwise, anything goes

\( n = 3 \)
$P = \text{ } n \times 2$

ex: $3 \times 2$

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1st row: when in time does $C_1$ fill in?

2nd row: " " " " $C_2$ " " " ?

$e^{\#}(n \times 2) = \frac{1}{n+1} \binom{2n}{n}$ famous **Catalan numbers**

$e(3 \times 2) = \frac{1}{4} \binom{6}{3} = 5 \checkmark$

View $e$ as function on $J(P)$

$e: J(P) \rightarrow \mathbb{Z} \in N$

$e(I) = \# \text{ extensions of } P$

$e(I) = \# \text{ extensions of } I$

then $e(I) = \sum e(I')$

I cover $I'$

generalizes Pascal's triangle
Example: $P = N + N$

$J(P) = N^2$

(number from 1, we're actually taking $P = P + P$)

got:

$e(i, j), I = J(P)$

generalized Pascal's Triangle

$L = J(P), e : L \rightarrow IP$

count $\pi$ / count lattice paths / satisfy recurrence.
3.6 Incidence Algebra of locally finite poset

(If we get to it)

(locally finite: intervals are finite, e.g. $\mathbb{N}^+$)

$\mathcal{I}(P, k) = \sum \text{ functions } f : \text{ intervals of } P \rightarrow k$

$I(P)$ we suppress $k$, usually take $\mathbb{C}$

$f = \sum_{x \leq y} f_{xy} [x, y]$

where $f_{xy} \in k$, $[x, y]$ formal symbol

$[x, y] \cdot [z, w] = \sum_{y=z} \delta_{0} \delta_{y \neq z} [x, w], y = z$

Convolution $fg$ defined by this rule, linearity.

Makes $\mathcal{I}(P)$ into a $k$-algebra.

if $P$ finite choose linearization $P \rightarrow \mathbb{N}$

$\begin{bmatrix}
    f_{11} & f_{12} & f_{13} & f_{14} \\
    f_{22} & f_{23} & f_{24} \\
    f_{33} & f_{34} \\
    f_{44}
\end{bmatrix}$

with holes, $x_i \neq x_j$

$I(P) \cong \text{ algebra}$

of these matrices
Prop 3.6.2 \( f \in \mathcal{I}(P) \) is left inverse/right inverse/inverse if \( f(x, x) \neq 0 \ \forall x \in P \) (exercise)

\[
\delta = \sum_{x \in P} \delta_{(x,x)} \quad \text{identity}
\]

\[ S \text{ zeta function defined by} \]

- Fill in *'s with 1's

\[ S(x, y) = \sum 1, \ \forall x \leq y \]

\[ S^2(x, y) \not\equiv \left( \sum_{x \leq z \leq y} [x,z][z,y] = \sum_{x \leq z \leq y} [x,y] \right) \]

\[ \# [x,y] \]

\[ = \# \text{ multichains} \quad x = x_0 \leq x_1 \leq x_2 = y \]

\[ S^K(x,y) = \# \text{ multichains} \quad x = x_0 \leq x_1 \leq \ldots \leq x_K = y \]

\[ = \# \text{ degree K-1 monoms in chain ring of } (x_1 y) \]

\[ = \text{ # degree K-1 monoms in chain ring of } \Delta(x_1 y) \]

\[
\sum_{m=0}^{\infty} \dim(S^m) t^m = \sum_{m=0}^{\infty} S^{m+1}(x_1 y) t^m = K[\Delta(P)]
\]
\[(S-1)(x,y) = \begin{cases} 1, & x < y \\ 0, & x = y \end{cases}\]

\[(S-1)^\eta(0,1) \text{ for } \eta(P) = \# \text{ linearizations of } P\]

\[(S-1)^K(x,y) = \# \text{ chains } x = x_0 < x_1 < \ldots < x_k = y\]

so 3.5 gives additional interps when \(P\) is dist lattice

\[(2-S)(x,y) = \begin{cases} 1, & x = y \\ -1, & x < y \end{cases}\]

2-S invertible,

**Thm** \((2-S)^{-1}(x,y) = \text{ total } \# \text{ chains (any } K)\)

\[x = x_0 < \ldots < x_k = y\]

\[l = \text{ length of longest chain in } [x,y]\]

\[\Rightarrow (S-1)^{l+1}(u,v) = 0 \quad \forall u,v \quad x \leq u \leq v \leq y\]

so \(\sum \frac{(S-1)^l}{l!} \sum \frac{(S-1)^l}{l!} \ldots \frac{(S-1)^l}{l!} (u,v)\)

\[= (1-(S-1)) \left[ \begin{array} \vdots \end{array} \right] (u,v)\]

\[= \left[ 1 - (S-1)^{l+1} \right] (u,v)\]

\[= S(u,v) \quad \text{identity in } I(P)\]

\[\text{Similar } \eta(x,y) = \sum 1 \text{ if covers } x\]

\[\text{else } \eta(x,y) = \eta(0,1)\]

\[(1-\eta)^{-1}(x,y) = \text{ total } \# \text{ maximal chains in } [x,y]\]