Second homework problems

Dave Bayer, Modern Algebra, September 30, 1998

Figure 1

Problems 1 through 5 all concern the group $G$ of symmetries of a square.

[1] The group $G$ of symmetries of the square has 8 elements: the identity, 3 rotations, and 4 flips. $G$ can be generated by two elements: a quarter-turn $a$, and a flip $b$. By numbering the square as shown in Figure 1, express the 8 elements of $G$ as permutations on \{1, 2, 3, 4\}. This expresses $G$ as a subgroup of the group $S_4$ of all permutations on \{1, 2, 3, 4\}. What permutations represent your choices of $a$ and $b$?

[2] In terms of generators and relations, $G$ can be written as the group

$$G = \langle a, b \mid a^4 = b^2 = 1, \ b a = a^m b \rangle$$

for some choice of $m$. What is $m$? Listing the elements of $G$ in the form

$$1, \ a, \ a^2, \ a^3, \ b, \ a b, \ a^2 b, \ a^3 b,$$

describe each of these elements both as symmetries of the square, and as permutations in $S_4$.

[3] Show that $S_4$ can be generated by the two permutations $c = (1\ 2)$ and $d = (1\ 2\ 3\ 4)$. How would you demonstrate this to a high school student, using props but no notation?

[4] Decide whether or not $G$ is a normal subgroup of $S_4$, using the criterion

$$G \subseteq S_4 \text{ is normal } \iff g G g^{-1} = G \text{ for all } g \in S_4.$$

Show that it is enough to check that $g h g^{-1} \in G$ for each generator $g$ of $S_4$ and each generator $h$ of $G$. Apply this to the generators $c, d$ of $S_4$ and $a, b$ of $G$ which you found above, checking

$$c a c^{-1} \in G? \quad c b c^{-1} \in G? \quad d a d^{-1} \in G? \quad d b d^{-1} \in G?$$

[5] Decide whether or not the subgroup $H = \{1, a, a^2, a^3\} \subseteq G$ generated by $a$ is normal in $G$, using the same method. Repeat, for the subgroup $H = \{1, b\} \subseteq G$ generated by $b$. 

1
The remaining problems are independent of each other.

[6] Let the group $G$ be given in terms of generators and relations as

$$G = \langle a, b, c \mid abc = abc = 1 \rangle.$$  

How many distinct elements does $G$ have? Your answer may surprise you. Can you give a simpler presentation of $G$, perhaps using fewer generators? Do you recognize $G$?

[7] Let the group $G$ be given in terms of generators and relations as

$$G = \langle a, b \mid a^3 = b^3 = 1, \ ba = a^2 b \rangle.$$  

How many distinct elements does $G$ have? Your answer may surprise you. Can you give a simpler presentation of $G$, perhaps using fewer generators?

[8] (challenging) Let $p$ and $q$ be prime numbers. For which integers $m, 1 \leq m < p$, does the presentation

$$G = \langle a, b \mid a^p = b^q = 1, \ ba = a^m b \rangle.$$  

describe a group with $pq$ distinct elements? For these values of $m$, when is the subgroup generated by $a$ normal? When is the subgroup generated by $b$ normal?