Nearby cycles over general bases

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Plan of the talk

The Milnor fibration

Let $f\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ be a germ of holomorphic function having an isolated critical point at 0.

Theorem (Milnor 1967)

- *For $\epsilon > 0$ small, and $0 < \eta \ll \epsilon$, the restriction of f to*

$$B_\epsilon \cap f^{-1}(D_\eta) \to D_\eta,$$

where $B_\epsilon \subset \mathbf{C}^{n+1}$ is the ball of radius ϵ centered at 0 and $D_\eta \subset \mathbf{C}$ is the disk of radius η centered at 0, induces a fibration over $D_\eta - \{0\}$.

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- *The fiber $M_t = f^{-1}(t) \cap B_\epsilon$ is homotopy equivalent to a bouquet of μ n-spheres $S^n \vee \cdots \vee S^n$, where μ is the Milnor number:*

$$\mu = \dim \mathbf{C}\{z_0, \ldots, z_n\}/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n).$$

The monodromy action

We have

$$\Phi^i := \mathrm{Coker}(H^i(\mathrm{pt}) \to H^i(M_t)) = \begin{cases} \mathbf{Z}^\mu & i = n \\ 0 & i \neq n. \end{cases}$$

Letting t turn around 0 gives the monodromy operator $T \in \mathrm{Aut}(\Phi^i)$.

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Conjecture (Milnor)

T is quasi-unipotent: the eigenvalues of T are roots of unity.

Grothendieck proved this using his theory of nearby and vanishing cycles.

Plan of the talk

1. The Milnor fibration
2. Nearby cycles over one-dimensional bases
   - Definition and functoriality
   - The quasi-semistable case
   - Constructibility and duality
3. Nearby cycles over general bases
   - Motivation
   - Definition and properties
   - Künneth formula and applications
   - Duality

Grothendieck's nearby and vanishing cycles

Grothendieck first mentioned vanishing cycles in a letter to Serre in 1964.

Given a family $X \to S$ over a one-dimensional base, Grothendieck (1967) constructed in SGA 7 the complex of vanishing cycles, a complex of sheaves measuring:

- on the one hand, the singularity of the family; and,
- on the other, the difference between $H^*(X_s)$ and $H^*(X_t)$.

He also constructed a closely related complex of sheaves, called the complex of nearby cycles.

Settings: étale or complex analytic. We will concentrate on the étale setting.

A dictionary

Let S be the spectrum of a Henselian discrete valuation ring.
For simplicity assume S strictly local (in other words, the closed point $s \in S$ is separably closed).

D_η: disk	S
$0 \in D_\eta$: the center	$s \in S$: the closed point
$D_\eta - \{0\}$: punctured disk	$\eta \in S$: the generic point
$t \in D_\eta - \{0\}$	$\bar{\eta}$: a separable closure of η
$\pi_1(D_\eta - \{0\}, t) \simeq \mathbf{Z}$: the fund. group	$I = \mathrm{Gal}(\bar{\eta}/\eta)$: the inertia group
local systems on $D_\eta - \{0\}$	sheaves on $\eta_{\text{ét}}$

We have a short exact sequence $1 \to P \to I \to \prod_{\ell \neq p} \mathbf{Z}_\ell(1) \to 1$.
The wild inertia group P is a pro-p-group, where p is the char. of s.

Nearby cycle functor $R\Psi$

Let $X \to S$ be a morphism of schemes. Consider Cartesian squares:

$$\begin{array}{ccccc} X_s & \xrightarrow{i} & X & \xleftarrow{j} & X_{\bar{\eta}} \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & S & \longleftarrow & \bar{\eta}. \end{array}$$

Let $\Lambda = \mathbf{Z}/m\mathbf{Z}$, m invertible on S (or $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, etc., ℓ invertible on S). We work with sheaves of Λ-modules in étale topoi. $D(X) := D(\mathrm{Shv}(X_{\text{ét}}, \Lambda))$.

For $K \in D^+(X_\eta)$,

$$R\Psi K := i^* Rj_*(K|_{X_{\bar{\eta}}}) \in D^+(X_s).$$

Equipped with an action of the inertia group I.

Vanishing cycle functor Φ

For $K \in D^+(X)$, distinguished triangle on X_s:

$$K|_{X_s} \to R\Psi(K|_{X_\eta}) \to \Phi(K) \to K|_{X_s}[1].$$

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For a geometric point $x \to X_s$, distinguished triangle

$$\begin{array}{ccccccc} K_x & \longrightarrow & (R\Psi K)_x & \longrightarrow & (\Phi K)_x & \longrightarrow & K_x[1]. \\ \| & & \| & & & & \\ R\Gamma(X_{(x)}, K) & \longrightarrow & R\Gamma(X_{(x)\bar{\eta}}, K) & & & & \end{array}$$

B_ϵ: Milnor ball	$X_{(x)}$: strict localization
M_t: Milnor fiber	$X_{(x)\bar{\eta}}$

Functoriality

Let $h\colon X \to Y$ be a morphism of schemes over S.

- For h smooth, the canonical map

$$h_s^* R\Psi_Y \to R\Psi_X h_\eta^*$$

is an isomorphism. In particular, $(\Phi_X \Lambda)_x = 0$ at smooth points x of X/S.

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- For h proper, the canonical map

$$Rh_{s*} R\Psi_X \to R\Psi_Y Rh_{\eta *}$$

  is an isomorphism. In particular, for X/S proper, long exact sequence:

$$\begin{array}{ccccccc} H^i(X_s, K) & \xrightarrow{\mathrm{sp}} & H^i(X_s, R\Psi K) & \longrightarrow & H^i(X_s, \Phi K) & \longrightarrow & H^{i+1}(X_s, K). \\ & & \| & & & & \\ & & H^i(X_{\bar{\eta}}, K) & & & & \end{array}$$

The quasi-semistable case

Assume X regular, flat and of finite type over S, X_η smooth and $(X_s)_{\mathrm{red}}$ is a divisor with normal crossings.

Theorem (Grothendieck, modulo absolute purity)

$$(R^q\Psi\Lambda)_x^P \simeq \Lambda[I_t/nI_t](-q) \otimes_{\mathbf{Z}} \wedge^q C,$$

where $x \to X_s$ *is a geometric point,* $C = \mathrm{Ker}((n_1, \dots, n_r)\colon \mathbf{Z}^r \to \mathbf{Z})$*. Here* $n_1, \dots, n_r$ *are the multiplicities of the branches of* X_s *passing through* x*, and* $n = \gcd(n_1, \dots, n_r)$*.*

Absolute purity was known then for $S/\mathbf{Q}$, and in general by Gabber 1994.

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Topological model for the tame Milnor fiber $X_{(x)\eta_t}$: p-prime homotopy fiber of the homomorphism

$$(S^1)^r \to S^1 \quad (x_1, \dots, x_r) \mapsto \prod_i x_i^{n_i}.$$

Milnor's conjecture

Corollary

In the quasi-semistable case, an open subgroup J of I acts trivially on $(R^q\Psi\Lambda)^P$.

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An analytic version of this + Hironaka's resolution of singularities $\Rightarrow$

Corollary (Milnor's conjecture)

Let $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a germ of holomorphic functions having an isolated critical point at 0. Then T acts quasi-unipotently on Φ^i.

Grothedieck's local monodromy theorem

Theorem

Let X_η be a scheme of finite type over η. There exists an open subgroup $J \subseteq I$ such that for all $i \in \mathbf{Z}$ and all $g \in J$, $(g-1)^{i+1} = 0$ on $H^i(X_{\bar{\eta}})$.

Grothendieck gave two proofs.

- Arithmetic proof of quasi-unipotence without bound $i+1$.
- Geometric proof modulo absolute purity (Gabber 1994) and resolution of singularities (which can be replaced by de Jong's alterations, Gabber-Illusie 2014). Uses $R^q\Psi\Lambda$ in the quasi-semistable case.

The bound $i+1$ ($i=1$) is crucial for Grothendieck's proof of the semistable reduction theorem for Abelian varieties.

Constructibility and duality

Assume X/S separated of finite type.

Theorem (Deligne 1974)

$R\Psi$ preserves bounded constructible complexes:

$$R\Psi\colon D^b_{\mathrm{cons}}(X_\eta) \to D^b_{\mathrm{cons}}(X_s).$$

Constructibility and duality

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Theorem (Gabber 1981)

$R\Psi$ commutes with duality: For $K \in D^b_{\mathrm{cons}}(X_\eta)$,

$$R\Psi D_{X_\eta} K \simeq D_{X_s} R\Psi K.$$

Corollary

$R\Psi$ preserves perverse sheaves.

Duality and Φ

Theorem (Beilinson 1987)

Φ *commutes with duality up to twist: For* $K \in D^b_{\mathrm{cons}}(X)$,

$$\Phi D_{X_\eta} K \simeq \tau^{-1} D_{X_s} \Phi K.$$

Here τ *is the Iwasawa twist: for* $L^P = 0$, $\tau^{-1} L = L$; *for* $L = L^P$,

$$\tau^{-1} L = \mathrm{Hom}^\times(I_t, L).$$

Proof uses Beilinson's maximal extension functor Ξ.

Sliced nearby cycle functor $R\Psi^s$ (Deligne)

Recall the distinguished triangle:

$$K|_{X_s} \to R\Psi(K|_{X_\eta}) \to \Phi(K) \to K|_{X_s}[1].$$

- $K|_{X_s}$ lives on X_s.
- $R\Psi(K|_{X_\eta})$ lives on the product topos $X_s \times \eta := (X_s)_{\text{ét}} \times \eta_{\text{ét}}$. A sheaf on $X_s \times \eta$ is a sheaf on X_s equipped with a continuous action of I.
- The two can be glued together to form $R\Psi^s(K)$ living on the product topos $X_s \times S := (X_s)\text{ét} \times S_{\text{ét}}$. A sheaf on $X_s \times S$ consists of a triple $(\mathcal{F}_s, \mathcal{F}_\eta, \text{sp})$ with $\mathcal{F}_s$ on X_s, $\mathcal{F}_\eta$ on $X_s \times \eta$, and $\text{sp}\colon p^*\mathcal{F}_s \to \mathcal{F}_\eta$.

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- Φ is the composition

$$D^+(X) \xrightarrow{R\Psi^s} D^+(X_s \times S) \xrightarrow{LC} D^+(X_s \times \eta),$$

where $C(\mathcal{F}_s, \mathcal{F}_\eta, \mathrm{sp}) = \mathrm{Coker}(\mathrm{sp})$.

$R\Psi^s$ and duality

$$\begin{array}{ccccc} D^+(X_\eta) & \longleftarrow & D^+(X) & & \\ {\scriptstyle R\Psi}\downarrow & & {\scriptstyle R\Psi^s}\downarrow & \overset{\Phi}{\searrow} & \\ D^+(X_s \times \eta) & \xleftarrow{j^*} & D^+(X_s \times S) & \xrightarrow{LC} & D^+(X_s \times \eta). \end{array}$$

Arrows $\leftarrow$ are restrictions.

$R\Psi^s$ and duality

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\downarrow{\scriptstyle R\Psi} & & \downarrow{\scriptstyle R\Psi^s} & \searrow^{\Phi} & \\
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Conjecture (Deligne 1999, letter to Illusie)

$R\Psi^s$ commutes with duality: $R\Psi^s D_X K \simeq D_{X_s \times S} R\Psi^s K$ *for* $K \in D^b_{\mathrm{cons}}(X)$.

$R\Psi^s$ and duality

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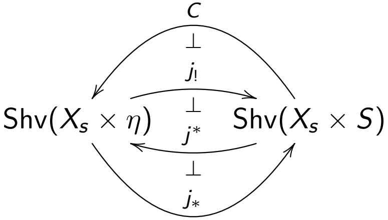
Theorem (Lu-Z. 2017)

- *Deligne's conjecture holds.*
- $(LC)D_{X_s\times S} \simeq \tau^{-1} D_{X_s\times\eta}(LC)$.

$\Rightarrow$ new proof of Beilinson's theorem $\Phi DK \simeq \tau^{-1} D\Phi K$ for $K \in D^b_{\mathrm{cons}}$.

LC and duality

Adjoint functors:



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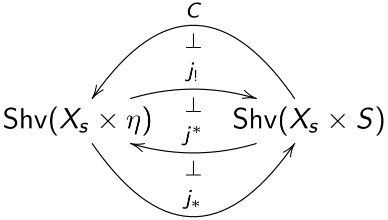
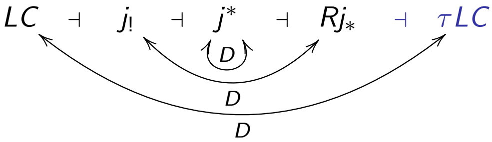


Adjoint functors between derived categories:



Plan of the talk

Motivation: the Sebastiani-Thom theorem

- Let $f_i\colon (\mathbf{C}^{n_i+1},0)\to(\mathbf{C},0)$, $i=1,2$ be germs of holomorphic functions with isolated critical point at 0.
- Define $f_1\oplus f_2\colon (\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$ by $(x_1,x_2)\mapsto f_1(x_1)+f_2(x_2)$, where $n=n_1+n_2+1$. It has isolated critical point at 0.

Theorem (Sebastiani-Thom 1971)

$$\Phi^{n_1}_{f_1}\otimes\Phi^{n_2}_{f_2}\simeq\Phi^{n+1}_{f_1\oplus f_2},$$
$$T_{f_1}\otimes T_{f_2}=T_{f_1\oplus f_2}.$$

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$$T_{f_1} \otimes T_{f_2} = T_{f_1 \oplus f_2}.$$

Deligne: An ℓ-adic analogue compatible with Galois action could not hold in characteristic > 0. Need to replace $\otimes$ by local convolution product $*$.

Sebastiani-Thom theorem in characteristic ≥ 0

Let k be an algebraically closed field.
Let $f_i\colon X_i \to \mathbf{A}^1_k$ be morphisms of schemes of finite type.
$f_1 \oplus f_2$ is the composition

$$X_1 \times_k X_2 \xrightarrow{f_1 \times_k f_2} \mathbf{A}^1_k \times_k \mathbf{A}^1_k \xrightarrow{+} \mathbf{A}^1_k.$$

Theorem (Deligne 1980, Fu 2014)

Assume X_i smooth over k of dimension $n_i + 1$, and f_i has isolated singularity at x_i. Then

$$\Phi^{n_1}_{f_1}(\Lambda)_{x_1} * \Phi^{n_2}_{f_2}(\Lambda)_{x_2} \simeq \Phi^{n_1+n_2+1}_{f_1 \oplus f_2}(\Lambda)_{(x_1,x_2)}.$$

A generalization

Suggested by Deligne 2011, letter to Fu.

Theorem (Illusie 2017)

(No assumptions on X_i or f_i.) For $K_i \in D^{\mathrm{ft}}_{\mathrm{cons}}(X_i)$,

$$R\Psi_{f_1}(K_1) *^L R\Psi_{f_2}(K_2) \simeq R\Psi_{f_1 \oplus f_2}(K_1 \boxtimes^L K_2).$$

Proof uses nearby cycles for $f_1 \times_k f_2 \colon X_1 \times_k X_2 \to \mathbf{A}^1_k \times_k \mathbf{A}^1_k$ over a two-dimensional base.

Oriented products of topoi (Deligne)

Deligne's nearby cycles over general bases live on vanishing topoi, which are a type of oriented products of topoi. Given morphisms of topoi $f\colon X \to S$ and $g\colon Y \to S$, the oriented product is a topos $X \overleftarrow{\times}_S Y$ together with a diagram

$$\begin{array}{ccccc} & & X \overleftarrow{\times}_S Y & & \\ & \swarrow & & \searrow & \\ X & & \Leftarrow & & Y \\ & {}_f\searrow & & \swarrow_g & \\ & & S, & & \end{array}$$

universal for these data.

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universal for these data.

Example

- The vanishing topos $X \overleftarrow{\times}_S S$.
- The covanishing topos $S \overleftarrow{\times}_S Y$. A generalization (Falting's topos) is used in p-adic comparison theorems.

Oriented product of topoi: Construction

Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be morphisms of schemes.

Site for $X \overleftarrow{\times}_S Y := X_{\text{ét}} \overleftarrow{\times}_{S_{\text{ét}}} Y_{\text{ét}}$:

- Objects: Commutative diagrams

$$\begin{array}{ccccc} U & \longrightarrow & W & \longleftarrow & V \\ \downarrow{\scriptstyle \text{ét.}} & & \downarrow{\scriptstyle \text{ét.}} & & \downarrow{\scriptstyle \text{ét.}} \\ X & \xrightarrow{f} & S & \xleftarrow{g} & Y. \end{array}$$

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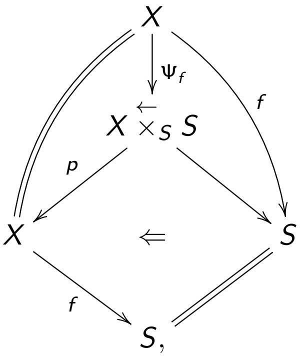
- Morphisms: Obvious.
- Covering families:
  - $(U_i \to W \leftarrow V)_{i\in I}$ above $U \to W \leftarrow V$ with $(U_i)_{i\in I}$ covering U;
  - $(U \to W \leftarrow V_i)_{i\in I}$ above $U \to W \leftarrow V$ with $(V_i)_{i\in I}$ covering V.
  -

$$\begin{array}{ccccc} U & \longrightarrow & W' & \longleftarrow & V' \\ \| & & \downarrow & \square & \downarrow \\ U & \longrightarrow & W & \longleftarrow & V. \end{array}$$

Nearby cycles over general bases (Deligne)

Let $f\colon X \to S$ be a morphism of schemes. Diagram of topoi:



For $K \in D^+(X)$, distinguished triangle in $D^+(X \overleftarrow{\times}_S S)$:

$$p^*K \to R\Psi_f K \to \Phi_f K \to p^*K[1].$$

Stalks

The points of $X \overleftarrow{\times}_S S$ are triples (x, t, sp), where $x \to X$, $t \to S$ are geometric points, $\mathrm{sp}\colon t \to S_{(f(x))}$ is a specialization.

$$(R\Psi_f K)_{(x,t)} = R\Gamma(X_{(x)} \times_{S_{f(x)}} S_{(t)}, K)$$

$X_{(x)} \times_{S_{f(x)}} S_{(t)}$ is the Milnor tube (containing the Milnor fiber $X_{(x)} \times_{S_{f(x)}} t$).

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$X_{(x)} \times_{S_{f(x)}} S_{(t)}$ is the Milnor tube (containing the Milnor fiber $X_{(x)} \times_{S_{f(x)}} t$).

Example

Assume S is the spectrum of a strictly local discrete valuation ring (one-dimensional). Then

$$X \overleftarrow{\times}_S S = X_\eta \cup (X_s \times \eta) \cup X_s.$$

$R\Psi_f K$ on these three shreds are $K|_{X_\eta}$, $R\Psi(K|_{X_\eta})$, $K|_{X_s}$, respectively.

A bad example

Let k be an algebraically closed field.
Let $S = \mathbf{A}^2_k$. Let $f\colon X := \mathrm{Bl}_O(S) \to S$ be blow-up at the origin O.
For geometric points $x \to X_O$, $t \to S_{(O)} - \{O\} \simeq X - X_O$, the Milnor tube is a join:

$$X_{(x)} \times_{S_{(O)}} S_{(t)} = X'_{(x)} \times_{X'} X'_{(t)},$$

which has infinitely many connected components. Here $X' = X \times_S S_{(O)}$.
Thus

$$(\Psi_f \Lambda)_{(x,t)} = H^0(X'_{(x)} \times_{X'} X'_{(t)}, \Lambda)$$

is not a finitely generated Λ-module.

By a theorem of M. Artin, $R\Psi_f\Lambda = \Psi_f\Lambda$.

Constructibility and base change

Let $X \to S$ be a morphism of finite type of Noetherian schemes. Let $K \in D^b_{\mathrm{cons}}(X)$.

Theorem (Orgogozo 2006)

- *There exists a modification $S' \to S$ such that $R\Psi_{f_{S'}}(K|_{X_{S'}})$ commutes with base change $T \to S'$.*
- *For S' as above, $R\Psi_{f_{S'}}(K|_{X_{S'}}) \in D^b_{\mathrm{cons}}$.*

Analytic analogue (Sabbah 1983):
Every morphism becomes "without blow-up" up to blowing up the base.

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Analytic analogue (Sabbah 1983):
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Corollary

Assume S regular of dimension 1. Then $R\Psi_f K$ commutes with base change $T \to S$.

Case $T \to S$ finite due to Deligne (with a gap found by Fu and fixed by Deligne in 1999).

Künneth formula for nearby cycles

Illusie's generalization of the Sebastiani-Thom theorem follows from the following Künneth formula.

Theorem (Illusie 2017)

Let $f_i \colon X_i \to Y_i$ be morphisms locally of finite type of schemes over a base scheme S.
Let $K_i \in D^b(X_i)$ such that $R\Psi_{f_i} K_i$ commutes with base change. Then

$$R\Psi_{f_1} K_1 \boxtimes^L R\Psi_{f_2} K_2 \simeq R\Psi_{f_1 \times_S f_2}(K_1 \boxtimes^L K_2).$$

Case $Y_1 = Y_2 = S$ of dimension 1 due to Gabber (1981).

Application: Global index formula (background)

Let k be an algebraically closed field. Let V be a variety over k.
Let $\mathcal{F}$ be a local system on V ($\Lambda = \mathbf{Z}/\ell\mathbf{Z}$ or $\mathbf{Q}_\ell$).

Theorem (Deligne)

If $\mathrm{char}(k) = 0$ *or more generally if* $\mathcal{F}$ *is tamely ramified at infinity, then*

$$\chi(V, \mathcal{F}) = \chi(V)\mathrm{rk}(\mathcal{F})$$

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Theorem (Grothendieck-Ogg-Shafarevich)

Let C *be a projective smooth curve over* k *and let* $V \subseteq C$ *be an open subset. Then*

$$\chi(V, \mathcal{F}) = \chi(V)\mathrm{rk}(\mathcal{F}) - \sum_{x \in C - V} \mathrm{Sw}_x(\mathcal{F}).$$

The Swan conductor $\mathrm{Sw}_x(\mathcal{F}) \in \mathbf{Z}_{\geq 0}$ measures the wild ramification of $\mathcal{F}$ at x.

Application: Global index formula

Let X be a smooth variety of dimension d over k. Let $\mathcal{F} \in \mathrm{Shv}_{\mathrm{cons}}(X)$.

- Beilinson (2016) defined the singular support $SS(\mathcal{F}) \subset T^*X$, a conic subset, equidimensional of dimension d. ("$\mathcal{F}$ is holonomic", but $SS(\mathcal{F})$ not Lagrangian in general.)
- T. Saito (2017) defined the characteristic cycle $CC(\mathcal{F})$, a d-cycle supported on $SS(\mathcal{F})$.

Theorem (T. Saito 2017)

Assume X projective.

$$\chi(X, \mathcal{F}) = (CC(\mathcal{F}), 0).$$

Inspired by Kashiwara-Dubson index formula (analytic setting) and conjectures of Deligne. Proof uses Künneth formula for nearby cycles.

Nearby cycles and duality

Let $f\colon X \to S$ be a separated morphism of finite type of excellent schemes.

Question (Illusie)

Does $R\Psi_f$ commute with duality?

Nearby cycles and duality

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Does $R\Psi_f$ commute with duality?

- One can define $D_{X \overleftarrow{\times}_S S}$ such that $R\Psi_f D_X \simeq D_{X \overleftarrow{\times}_S S} R\Psi_f$.
- No reasonable duality on $X \overleftarrow{\times}_S S$ even if $\dim(S) = 1$.

Sliced nearby cycles and duality

For any geometric point $s \to S$, $X_s \overleftarrow{\times}_S S \simeq X_s \times S_{(s)}$.

Definition (Sliced nearby cycles)

$$R\Psi_f^s K := (R\Psi_f K)|_{X_s \times S_{(s)}}.$$

Theorem (Lu-Z. 2017)

Assume S finite-dimensional. Let $K \in D^b_{\text{cons}}(X)$. Sliced nearby cycles commute with duality up to modification: There exists a modification $S' \to S$ such that for every morphism $T \to S'$ separated of finite type and every geometric point $t \to T$,

$$R\Psi_{f_T}^t D_{X_T}(K|_{X_T}) \simeq D_{X_t \times T_{(t)}} R\Psi_f^t (K|_{X_T}).$$

Application to local acyclicity

Corollary

Assume S regular. Then (f, K) is universally locally acyclic if and only if $(f, D_X K)$ is universally locally acyclic.

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Assume S regular. Then (f, K) is universally locally acyclic if and only if $(f, D_X K)$ is universally locally acyclic.

Theorem (Gabber)

Let $f\colon X \to S$ be a morphism of finite type of Noetherian schemes. If (f, K) is locally acyclic, then it is universally locally acyclic.

This answers a question of M. Artin in SGA 4.

The end

Thank you!

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