Homework 2, Representations of Finite Groups
Due 10/18/06

Suppose we are given an action F of a finite group G on a set X of n elements, i.e. a group homomorphism of G into the group of automorphisms of X:

$$F : G \to \mathrm{Aut}(X).$$

We define the associated *permutation representation* of G in the following way. Let V be a vector space of dimension n with a basis e_x indexed by the elements of X. Then the representation is defined on this basis as

$$\rho_F : G \to \mathrm{GL}(V), \quad \rho_F(g)(e_x) = e_{F(g)(x)}.$$

The natural representation of $\mathfrak{S}_n$ on $\mathbb{C}^n$ and the regular representation are examples of this.

1. Prove the following *fixed-point formula*: If V is the permutation representation associated to the action F of a finite group G on a finite set X, then $\chi_V(g)$ is equal to the number of elements of X fixed by $F(g)$.

2. We have seen that $\mathfrak{S}_4$ is the group of symmetries of the tetrahedron. Since $\mathfrak{S}_4$ acts on the set of vertices of the tetrahedron, it also acts on the set of its edges. Decompose the corresponding permutation representation into a sum of irreducibles.

3. Do the same for the permutation representation associated to the action of $\mathfrak{S}_4$ on the set of faces of the tetrahedron.

4. We have also seen that $\mathfrak{S}_4$ is the group of rotations of the cube. Therefore, we have an associated action of $\mathfrak{S}_4$ on the set of vertices of the cube. Decompose the corresponding permutation representation into a sum of irreducibles.

5. Do the same for the permutation representation associated to the action $\mathfrak{S}_4$ on the set of edges of the cube.

6. Do the same for the permutation representation associated to the action $\mathfrak{S}_4$ on the set of faces of the cube.