# $p$-adic Hodge Theory (Spring 2023): Week 9

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**This week: de Rham Representations**

## 1 Formal properties of admissible representations

fix a $(\mathbb{Q}_p, \Gamma_K)$-regular ring $B$ and write $E := B^{\Gamma_K}$.

**Theorem 1.1.** *For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ we have the following statements:*

1. *The natural map*
$$\alpha_V : D_B(V) \otimes_E B \to V \otimes_{\mathbb{Q}_p} B$$
*is $B$-linear, $\Gamma_K$-equivariant, and injective.*

2. *We have an inequality*
$$\dim_E D_B(V) \leq \dim_{\mathbb{Q}_p} V \tag{1}$$
*with equality if and only if $\alpha_V$ is an isomorphism.*

*Proof.* The natural $\alpha_V$ is given by

$$\alpha_V : D_B(V) \otimes_E B \to (V \otimes_{\mathbb{Q}_p} B) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} (B \otimes_E B) \to V \otimes_{\mathbb{Q}_p} B,$$

which is $B$-linear and $\Gamma_K$-equivariant by inspection. We need to show that $\alpha_V$ is injective. The fraction field $C$ of $B$ is $(\mathbb{Q}_p, \Gamma_K)$-regular. We thus have a natural map

$$\beta_V : D_C(V) \otimes_E C \to V \otimes_{\mathbb{Q}_p} C$$

which fits into a commutative diagram

$$\begin{array}{ccc} D_B(V) \otimes_E B & \xrightarrow{\alpha_V} & V \otimes_{\mathbb{Q}_p} B \\ \big\downarrow & & \big\downarrow \\ D_C(V) \otimes_E C & \xrightarrow{\beta_V} & V \otimes_{\mathbb{Q}_p} C \end{array}$$

where both vertical maps are injective. Therefore it suffices to prove the injectivity of $\beta_V$.
Let $(x_i)$ be a basis of $D_C(V) = (V \otimes_{\mathbb{Q}_p} C)^{\Gamma_K}$ over $E$. We regard each $x_i$ as an element in $V \otimes_{\mathbb{Q}_p} C$. Note that $(x_i)$ spans $D_C(V) \otimes_E C$ over $C$.
Assume for contradiction that the kernel of $\beta_V$ is not trivial. Then we have a nontrivial relation of the form $\sum b_i x_i = 0$ with $b_i \in C$. Let us choose such a relation with minimal length. We may assume $b_r = 1$ for some $r$. For every $\gamma \in \Gamma_K$ we find

$$0 = \gamma(\sum b_i x_i) - \sum b_i x_i = \sum (\gamma(b_i) - b_i) x_i.$$

Since the coefficient of $x_r$ vanishes, the minimality of our relation yields $b_i = \gamma(b_i)$ for each $b_i$, or equivalently $b_i \in C^{\Gamma_K} = E$. Hence our relation gives a nontrivial relation for $(x_i)$ over $E$, thereby yielding a desired contradiction.
Since the extension of scalars from $B$ to $C$ preserves injectivity, $\alpha_V$ induces an injective map

$$D_B(V) \otimes_E C \hookrightarrow V \otimes_{\mathbb{Q}_p} C. \tag{2}$$

The desired inequality now follows by observing

$$\dim_C D_B(V) \otimes_E C = \dim_E D_B(V) \quad \text{and} \quad \dim_C V \otimes_{\mathbb{Q}_p} C = \dim_{\mathbb{Q}_p} V. \tag{3}$$

Hence it remains to consider the equality condition. If $\alpha_V$ is an isomorphism, the map (2) also becomes an isomorphism, thereby yielding equality in (1) by (3). Let us now assume that equality in (1) holds, and write

$$d := \dim_E D_B(V) = \dim_{\mathbb{Q}_p} V.$$

By (3) we find that the map (2) is an isomorphism for being an injective map between two vector spaces of the same dimension. Let us choose a basis $(e_i)$ of $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ over $E$ and a basis $(v_i)$ of $V$ over $\mathbb{Q}_p$. Then we can represent $\alpha_V$ by a $d \times d$ matrix $M_V$ . We have $\det(M_V \neq 0$ as $\alpha_V$ induces an isomorphism (2). We wish to show $\det(M_V) \in B^\times$.
Let us consider the identity

$$\alpha_V(e_1 \wedge \cdots \wedge e_d) = \det(M_V)(v_1 \wedge \cdots \wedge vd).$$

By construction, $\Gamma_K$ acts trivially on $e_1 \wedge \cdots \wedge e_d$ and by some $\mathbb{Q}_p$-valued character $\eta$ on $v_1 \wedge \cdots \wedge v_d$. Since $\alpha_V$ is $\Gamma_K$-equivariant, we deduce that $\Gamma_K$ acts on $\det(M_V)$ by $\eta^{-1}$. Hence we obtain $\det(M_V) \in B^\times$ as $B$ is $(\mathbb{Q}_p, \Gamma_K)$-regular, thereby completing the proof. □

**Proposition 1.2.** *The functor $D_B$ is exact and faithful on $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$.*

*Proof.* Let $V$ and $W$ be $B$-admissible representations. Suppose that $f \in \mathrm{Hom}_{\mathbb{Q}_p}[\Gamma_K](V, W)$ induces a zero map $D_B(V) \to D_B(W)$. Then $f$ induces a zero map $V \otimes_{\mathbb{Q}_p} B \to W \otimes_{\mathbb{Q}_p} B$ by Theorem 1.1, which means that $f$ must be a zero map. We thus find that the functor $D_B$ is faithful on $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$. It remains to verify that $D_B$ is exact on $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$. Let us consider an arbitrary short exact sequence of B-admissible representations

$$0 \to U \to V \to W \to 0.$$

Recall that every algebra over a field is faithfully flat; in particular, $B$ is faithfully flat over both $\mathbb{Q}_p$ and $E$. Therefore we find that the sequence

$$0 \to U \otimes_{\mathbb{Q}_p} B \to V \otimes_{\mathbb{Q}_p} B \to W \otimes_{\mathbb{Q}_p} B \to 0.$$

is exact, which implies that the sequence

$$0 \to D_B(U) \otimes_E B \to D_B(V) \otimes_E B \to D_B(W) \otimes_E B \to 0$$

is also exact by Theorem 1.1. The desired assertion now follows by the fact that $B$ is faithfully flat over $E$. □

**Proposition 1.3.** *The category $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ is closed under taking subquotients.*

*Proof.* Consider a short exact sequence of p-adic representations

$$0 \to U \to V \to W \to 0$$

with $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$. We wish to show that both $U$ and $W$ are $B$-admissible. SInce the functor $D_B$ is left exact by consturction, we have a left exact seuqence

$$0 \to D_B(U) \to D_B(V) \to D_B(W).$$

In addition, by Theorem 1.2.1 we have inequalities

$$\dim_E D_B(U) \leq \dim_{\mathbb{Q}_p} U \quad \text{and} \quad \dim_E D_B(W) \leq \dim_{\mathbb{Q}_p} W.$$

Then the exact sequences gives us

$$\dim_E D_B(V) \leq \dim_E D_B(U) + \dim_E D_B(W) \leq \dim_{\mathbb{Q}_p} U + \dim_{\mathbb{Q}_p} W = \dim_{\mathbb{Q}_p} V,$$

which are in fact equalities as $V$ is $B$-admissible. We thus have equalities in above, thereby deducing the desired assertion. □

In general, the category of $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ is not closed under taking extensions. In fact, there is an example which is Hodge–Tate but not de Rham given any non-split extension $V$:

$$0 \to \mathbb{Q}_p \to V \to \mathbb{Q}_p(1) \to 0$$

We also note the following proposition, for which we will skip the proof, but Serin's notes contains a good coverage of the proof.

**Proposition 1.4.** *Give $V, W \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, we have $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with a natural isomorphism*

$$D_B(V) \otimes_E D_B(W) \cong D_B(V \otimes_{\mathbb{Q}_p} W).$$

**Proposition 1.5.** *For every $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, we have $\wedge^n(V) \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ and that $\mathrm{Sym}^n V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with natural isomorphisms*

$$\wedge^n(D_B(V)) \cong D_V(\wedge^n(V)) \quad \text{and} \quad \mathrm{Sym}^n(D_B(V)) \cong D_B(\mathrm{Sym}^n(V)).$$

*Proof.* Let us only consider exterior powers here, as the same argument works with symmetric powers. By Proposition 1.4 we have $V^{\otimes n} \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with a natural isomorphism $D_B(V^{\otimes n}) \cong D_B(V)^{\otimes n}$. Hence by Proposition 1.3 we have $\wedge^n(V) \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with a natural $E$-linear map

$$0 \to D_B(V)^{\otimes n} \xrightarrow{\sim} D_B(V^{\otimes n}) \twoheadrightarrow D_B(\wedge^n(V))$$

where the surjectivity of the second arrow follows from the exactness of $D_B$ as noted in Proposition 1.2. It is then straightforward to check that this map factors through the natural surjection $D_B(V)^{\otimes n} \twoheadrightarrow \wedge^n(D_B(V))$. We thus obtain a natural surjective $E$-linear map

$$\wedge^n(D_B(V)) \twoheadrightarrow D_V(\wedge^n(V)),$$

which turns out to be an isomoprhism since we have

$$\dim_E \wedge^n(D_B(V)) = \dim_E D_B(\wedge^n(V))$$

by the $B$-admissibility of $V$ and $\wedge^n(V)$. □

**Proposition 1.6.** *If $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, $V^\vee \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with a perfect pairing:*

$$D_B(V) \otimes_E D_B(V^\vee) \xrightarrow{\cong} D_B(V \otimes_{\mathbb{Q}_p} V^\vee) \cong D_B(\mathbb{Q}_p) = E$$

*Proof.* **Case 1:** $\dim_{\mathbb{Q}_p} V = 1$. We want to show $\dim_E D_B(V^\vee) = 1 = \dim_{\mathbb{Q}_p} V^\vee$. Choose a basis $v$ of $V$ over $\mathbb{Q}_p$. There exists a character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ such that

$$\gamma(v) = \eta(\gamma)v \quad \text{for all } \gamma \in \Gamma_K.$$

Since $V$ is $B$-admissible, $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ is 1-dimensional. Hence, there exists $b \in B$ such that $v \otimes b$ is a $\Gamma_K$-invariant $E$-basis of $D_B(V)$.
Since $V$ is $B$-admissible, then the map $\alpha_V : D_B(V) \otimes_E B \xrightarrow{\cong} V \otimes_{\mathbb{Q}_p} B$ is an isomorphism, and hence it maps $v \otimes b$ to a basis $V \otimes_{\mathbb{Q}_p} B$. Hence $b \in B^\times$. Finally, we note that

$$\gamma(v \otimes b) = \gamma(v) \otimes \gamma(b) = \eta(\gamma)v \otimes \gamma(b) = v \otimes \eta(\gamma)\gamma(b) \quad \text{for every } \gamma \in \Gamma_K,$$

Hence $b = \eta(\gamma)\gamma(b)$ for all $\gamma \in \Gamma_K$. This shows that

$$D_B(V^\vee) = (V^\vee \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$$

contains a non-zero $v^\vee \otimes b^{-1}$ where $v^\vee$ is a dual basis. Hence $V^\vee$ is $B$-admissible and $D_B(V^\vee)$ is spanned by $v^\vee \otimes b^{-1}$. One easily checks that hte pairing is perfect.

**Case 2.** General case. Let $d = \dim_{\mathbb{Q}_p} V$. There is a natural $\Gamma_K$-equivalent isomorphism

$$\Phi\colon \underbrace{\det(V^\vee)}_{\wedge^d V^\vee} \otimes \bigwedge^{d-1} V \cong V^\vee$$

given by

$$(f_1 \wedge \cdots \wedge f_d) \otimes (w_2 \wedge \cdots \wedge w_d) \mapsto (w_1 \mapsto \det(f_i(w_j))).$$

Since $V$ is $B$-admissible, $det(V) = \bigwedge^d V$ is $B$-admissible, hence

$$\det(V^\vee) = \det(V)^\vee$$

is $B$-admissible by case 1.

Since $\bigwedge^{d-1} V$ is $B$-admissible by Proposition 1.5, this shows that $V^\vee$ is also $B$-admissible. We finally want to show that the pairing is perfect.

Fact: If $W, W'$ are vector spaces with $d = \dim_E W = \dim_E W'$ then $W \times W' \to E$ is perfect if and only if

$$\det(W) \times \det(W') \to E$$

is perfect. But we have the induced pairing:

$$\begin{array}{ccc} \det(D_B(V)) \otimes \det(D_B(V^\vee)) & \longrightarrow & E \\ \downarrow = & & \downarrow = \\ D_B(\det(V)) \otimes D_B(\det(V^\vee)) & \longrightarrow & E \end{array}$$

Since $\dim \det(V) = 1$, this completes the proof. ☐

# 2 De Rham representations

The goal is to define and study:

1. The de Rham period ring $B_{dR}$.
2. de Rham representations.

The references for this section are [BC09, Sections 4, 6] and [Sch12]. **Outline of consturction of $B_{dR}$**

The field $\mathbb{C}_K$ is perfectoid. Hence $F = \mathbb{C}_K^\flat$ is a perfectoid field of characteristic $p$. Let $\mathcal{O}_F$ be the valuation ring of $F$.
We get a surjective ring homomorphism:

$$\theta : W(\mathcal{O}_F) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$$

which gives

$$\theta : W(\mathcal{O}_F)[1/p] \twoheadrightarrow \mathbb{C}_K$$

and we may consider $\ker(\theta)$. Then

$$B_{dR}^+ = \varprojlim_j W(\mathcal{O}_F)[1/p]/(\ker\theta)^j$$

$$B_{dR} = \mathrm{Frac}(B_{dR}^+)$$

# 3 Perfectoid fields and tilting.

**Definition 3.1.** *Let $C$ be a complete non-archimedean field of residue characteristic $p$ with valuation ring $\mathcal{O}_C$. Then $C$ is a perfectoid field if:*

1. *the valuation on $C$ is non-discrete,*
2. *the pth power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective.*

**Lemma 3.2.** *Let $C$ be a complete non-archimedean field of residue characteristic $p$ with non-trivial valuation. Assume that the pth power map is surjective on $C$. Then $C$ is perfectoid.*

*Proof.* We first check property (1). Let $v$ be the valuation on $C$ and suppose $v$ is discrete. Then there exists $x \in C$ with minimal positive valuation. Also, $x = y^p$ for some $y \in C$ by the surjecitivity of the $p$th power map.

Then

$$0 < v(y) = \frac{1}{p} v(x) < v(x)$$

which is a contradiction.

For (2), it suffices to show surjectivity on $\mathcal{O}_C$. For all $x \in \mathcal{O}_C$, there exists $y \in C$ such that $x = y^p$. Then $v(y) = \frac{1}{p}v(x) > 0$, so $y \in \mathcal{O}_C$. ☐

**Proposition 3.3.** *The field $\mathbb{C}_K$ is perfectoid.*

*Proof.* This follows from Lemma 3.2, since $\mathbb{C}_K$ is algebraically closed. ☐

**Proposition 3.4.** *A non-archimedean field of characteristic p is perfectoid if and only if it is complete and perfect.*

Fix a perfectoid field $C$. Write $\mathcal{O}_C$ for the valuation ring of $C$ and $v$ for the valuation on $C$.

**Definition 3.5.** *The tilt of $C$ is*

$$C^\flat = \varprojlim_{x \mapsto x^p} C$$

*with the natural multiplication.*

A priori, $C^\flat$ is a multiplicative monoid. We will later define a topology on it, which turns out to be equivalent to the inverse limit topology. We want to show $C^\flat$ is a perfectoid field of characteristic $p$.

**Lemma 3.6.** *Fix $\varphi \in C^\times$ such that $0 < v(\varphi) \leq v(p)$. For all $x, y \in \mathcal{O}_C$ with $x - y \in \varphi\mathcal{O}_C$, then*

$$x^{p^n} - y^{p^n} \in \varphi^{n+1}\mathcal{O}_C.$$

*Proof.* By the inequality, $\varphi$ divides $p$ in $\mathcal{O}_C$. We have that

$$x^{p^n} - y^{p^n} = (y^{p^{n-1}} - (y^{p^{n-1}} - x^{p^{n-1}}))^p - y^{p^n}$$

which shows the result by induction. ☐

In practice, if $C$ has characteristic 0, then we may choose $\varpi = p$. If $C$ has characteristic $p, C^\flat \cong C$, so in practice, we might as well assume $C$ has characteristic 0.

**Proposition 3.7.** *Fix $\varphi \in C^\times$ such that $0 < v(\varpi) \leq v(p)$. Then we have a multiplicative bijection:*

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_C \to \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$$

*induced by $\mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/\varpi\mathcal{O}_C$.*

*Proof.* The map is clearly multiplicative, so we only need to construct an inverse. Define

$$\ell : \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C \to \varprojlim_{x \mapsto x^p} \mathcal{O}_C$$

by setting for $\bar{c} = (\bar{c}_n) \in \varprojlim \mathcal{O}_C/\varpi\mathcal{O}_C$ for $\bar{c}_n \in \mathcal{O}_C/\varpi\mathcal{O}_C$: $\ell(\bar{c}) = (\ell_n(\bar{c}))$ and that $(\ell_n(\bar{c})) = \lim_{m\to\infty} c_{n+m}^{p^m}$, where $c_n \in \mathcal{O}_C$ lifts $\bar{c}_n$.

For $\ell, m, n \gg 0$,

$$c_{n+m+\ell}^{p^\ell} - c_{n+m} \in \varpi\mathcal{O}_C,$$

because

$$\overline{c_{n+m+\ell}^{p^\ell}} - \overline{c_{n+m}} = \overline{c_{n+m}} - \overline{c_{m+n}} = 0.$$

Hence Lemma 3.6 shows that

$$c_{n+m+\ell}^{p^{\ell+m}} - c_{n+m}^{p^m} \in \varpi^{m+1}\mathcal{O}_C.$$

Therefore, for all $n, (c_{n+m}^{p^m})$ is a Cauchy sequence in $\mathcal{O}_C$. Therefore,

$$\lim_{m\to\infty} c_{n+m}^{p^m} \text{ exists.}$$

To check $\ell$ is well-defined, choose another lift $c'_n$ of $c_n$. Then

$$c_n - c'_n \in \varpi\mathcal{O}_C,$$

so lemma 3.6 implies that

$$c_{n+m}^{p^m} - c'^{p^m}_{n+m} \in \varpi^{m+1}\mathcal{O}_C.$$

Hence the limit does not depend on the choice. Finally, we need to show that $\ell$ is inverse to the reduction map in the statement. We have that:

$$(c_n) \mapsto (\overline{c_n}) \mapsto (\lim_{m\to\infty} c_{n+m}^{p^m}) = (\lim_{n\to\infty} c_n) = (c_n),$$

$$(\overline{c_n}) \mapsto (\lim_{m\to\infty} c_{n+m}^{p^m}) \mapsto (\lim_{m\to\infty} \overline{c_{n+m}^{p^m}}) = (\lim_{m\to\infty} \overline{c_n}) = (\overline{c_n}).$$

showing that $\ell$ is the inverse. □

**Proposition 3.8.** *The tilt $C^\flat$ of $C$ is naturally a complete valued field of characteristic $p$ with the valuation $\nu^\flat$ given by $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in C^\flat$. Moreover, the valuation ring of $C^\flat$ is given by*

$$\mathcal{O}_{C^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_C$$

*Proof.* Fix an element $\varpi \in C^\times$ with $0 < \nu(\varpi) \le \nu(p)$. The ring $\mathcal{O}_C/\varpi\mathcal{O}_C$ is of characteristic $p$ since $\varpi$ divides $p$ in $\mathcal{O}_C$ by construction. Hence the ring structure on $\mathcal{O}_C/\varpi\mathcal{O}_C$ induces a natural ring structure on $\varinjlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$, which in turn yields a ring structure on

$$\mathcal{O} := \varprojlim_{x\mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$$

where the isomorphism is given by Proposition 3.7. Moreover, this ring structure on $\mathcal{O}$ does not depend on the choice of $\varpi$; indeed, by the proof of Proposition 3.7 we find that the sum of two arbitrary elements $a = (a_n)$ and $b = (b_n)$ in $\mathcal{O}$ is given by

$$(a+b)_n = \lim_{m\to\infty} (a_{m+n} + b_{m+n})^{p^m}.$$

We then identify $C^\flat$ as the fraction field of $\mathcal{O}$. It is clear by construction that $C^\flat$ is perfect of characteristic $p$.
We assert that $C^\flat$ admits a valuation $\nu^\flat$ given by $\nu^\flat(c) := \nu(c^\sharp)$ for every $c \in c^\flat$. It is evident by construction that $\nu^\flat$ is a multiplicative homomorphism. Let us now consider arbitrary elements $a = (a_n)$ and $b = (b_n)$ in $c^\flat$. We wish to establish an inequality

$$\nu^\flat(a+b) \ge \min(\nu^\flat(a), \nu^\flat(b)).$$

May assume $\nu^\flat(a) \ge \nu^\flat(b)$, equivalently $\nu(a_0) \ge \nu(b_0)$. Then for each $n \ge 0$ we have

$$\nu^\flat(a+b) = \nu^\flat((r+1)b) = \nu^\flat(r+1) + \nu^\flat(b) \ge \nu^\flat(b) = \min(\nu^\flat(a), \nu^\flat(b))$$

where the inequality follows by observing $r+1 \in \mathcal{O}$.

Let us now take an arbitrary element $c = (c_n) \in C^\flat$. We have an inequality

$$\nu(c_n) = \frac{1}{p^n}\nu(c_0) = \frac{1}{p^n}\nu^\flat(c) \quad \text{for each } n \ge 0.$$

Hence we deduce that $\mathcal{O}$ is indeed the valuation ring of $C^\flat$. Moreover, given any $N > 0$ the inequality above implies that we have $\nu(c_n) \ge \nu(\varpi)$ for all $n \le N$ if and only if $\nu^\flat(c) \ge p^N\nu(\varpi)$. Therefore the bijection $\mathcal{O} := \varprojlim_{x\mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ becomes a homeomorphism if we endow $\mathcal{O}_{C^\flat} = \mathcal{O}$ and $\varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ respectively with the $\nu^\flat$-adic topology and the inverse limit topology. As the latter topology is complete, it follows that $C^\flat$ is complete.

□

**Remark 3.9.** *Proof remains valid if $C$ is replaced by complete nonarchimedean field $L$. But $L$ not perfectoid then $L^\flat$ becomes trivial.*

**Proposition 3.10.** *The map $\mathcal{O}_{C^\flat} \to \mathcal{O}_C/p\mathcal{O}_C$ which sends each $c \in \mathcal{O}_{C^\flat}$ to the image of $c^\flat$ in $\mathcal{O}_C/p\mathcal{O}_C$ is a ring homomorphism.*

**Lemma 3.11.** *For every $y \in \mathcal{O}_C$ there exists an element $z \in \mathcal{O}_{C^\flat}$ with $y - z^\sharp \in p\mathcal{O}_C$.*

**Proposition 3.12.** *The valued fields $C$ and $C^\flat$ have the same value groups.*

*Proof.* Let $\nu^\flat$ be the valuation on $C^\flat$ given by $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in C^\flat$. Since we have $\nu^\flat((C^\flat)^\times)$ by construction, we only need to show $\nu(C^\times) \subseteq \nu^\flat((C^\flat)^\times)$. Let us consider an arbitrary element $y \in C^\times$. We wish to find an element $z \in (C^\flat)^\times$ with $\nu^\flat(z) = \nu(y)$. Since $\nu$ is nondiscrete, we can choose an element $\varpi \in \mathcal{O}_C$ with $0 < \nu(\varpi) < \nu(p)$.

Let us write $y = \varpi^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_C$ with $\nu(u) < \mu(\varpi)$. By Lemma 3.11 there exist elements $\varpi^\flat$ and $u^\flat$ in $\mathcal{O}_{C^\flat}$ with $\varpi - (\varpi^\flat)^\sharp \in p\mathcal{O}_C$ and $u - (u^\flat)^\sharp \in p\mathcal{O}_C$. Then we find

$$\nu^\flat(\varpi^\flat) = \nu((\varpi^\flat)^\sharp) = \nu\left((\varpi) - (\varpi - (\varpi^\flat)^\sharp)\right) = \nu(\varpi),$$

$$\varpi^\flat(u^\flat) = \nu((u^\flat)^\sharp) = \nu\left((u) - (u - (u^\flat)^\sharp)\right) = \nu(u).$$

Hence we obtain the desired assertion by taking $z = (\varpi^\flat)^n u^\flat$. □

**Corollary 3.13.** *The field $C^\flat$ is a perfectoid field of characteristic $p$.*

**Corollary 3.14.** *If $C$ is of characteristic $p$, there exists a natural identification $C^\flat \cong C$.*

**Example 3.15.** *Let $\mathbb{Q}_p\widehat{(p^{1/p^\infty}})$ denote the p-adic completions of $\bigcup_{n\geq 1} \mathbb{Q}_p(p^{1/p^n})$. The p-adic valuation of $\mathbb{Q}_p\widehat{(p^{1/p^\infty}})$ is not descrete. In addition, the valuation ring of $\mathbb{Q}_p\widehat{(p^{1/p^\infty}})$ is $\mathbb{Z}_p\widehat{[p^{1/p^\infty}}]$, the p=adic completion of $\mathbb{Z}_p$-algebra obtained by adjoining all p=th power roots of p. We also have an isomorphism*

$$\mathbb{Z}_p\widehat{[p^{1/p^\infty}}] \cong \mathbb{Z}_p[p^{1/p^\infty}]/p \cong \mathbb{F}_p[u^{1/p^\infty}]/u$$

*where $\mathbb{F}_p[u^{1/p^\infty}]$ denotes the perfection of the polynomial ring $\mathbb{F}_p[u]$. Since the p-th power map on $\mathbb{F}_p[u^{1/p^\infty}]/u$ is surjective, we deduce $\mathbb{Q}_p\widehat{(p^{1/p^\infty}})$ is perfectoid.*

$$\lim_{x\mapsto x^p} \mathbb{Z}_p\widehat{[p^{1/p^\infty}}] \cong \lim_{x\mapsto x^p} \mathbb{F}_p[u^{1/p^\infty}]/u \cong \mathbb{F}_p\widehat{[u^{1/p^\infty}}]$$

*where $\mathbb{F}_p\widehat{[u^{1/p^\infty}}]$ denotes the u-adic compleition of $\mathbb{F}_p[u^{1/p^\infty}]$, and hence $\mathbb{Q}_p\widehat{(p^{1/p^\infty}})$ is isomorphic to $\mathbb{F}_p\widehat{((u^{1/p^\infty}}))$, the u-adic completion of the operfection of the Laurent series ring $\mathbb{F}_p((u))$.*

# 4 De Rham Period Ring $B_{dR}$

Write $F := C_K^\flat$ for the tilt of $\mathbb{C}_K$. In addition, for every element $c = (c_n)_{n\geq 0}$ in $F$ we write $c^\sharp := c_0$. We also fix a valuation $\nu$ on $\mathbb{C}_K$ with $\nu(p) = 1$, and let $\nu^\flat$ denote the valuation on $F$ given by $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in F$.

**Definition 4.1.** *We define the infinitesimal period ring, denoted by $A_{inf}$, to be the ring of Witt vectors over $\mathcal{O}_F$ . For every $c \in \mathcal{O}_F$ , we write $[c]$ for its Teichmuller lift in $A_{inf}$.*

Note that ring $A_{inf}$ is not $(\mathbb{Q}_p, \Gamma_K)$-regular in any meaningful way.

**Proposition 4.2.** *There exists a surjective ring homomorphism $\theta : A_{inf} \to \mathcal{O}_{\mathbb{C}_K}$, with*

$$\theta\left(\sum_{n=0}^{\infty}[c_n]p^n\right) = \sum_{n=0}^{\infty} c_n^\sharp p^n \quad \textit{for all } c_n \in \mathcal{O}_F.$$

*Proof.* Let us define a map $\overline{\theta} : \mathcal{O}_F \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ by

$$\overline{\theta}(c) = \overline{c^\sharp} \quad \text{for every } c \in \mathcal{O}_F$$

where $\overline{c^\sharp}$ denotes the image of $c^\sharp$ in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Then $\overline{\theta}$ is a ring homomorphism as noted in proposition 3.10. Moreover, by construction $\overline{\theta}$ lifts to a map $\widehat{\theta} : \mathcal{O}_F \to \mathcal{O}_{\mathbb{C}_K}$ by

$$\widehat{\theta}(c) = c^\sharp \quad \text{for every } c \in \mathcal{O}_F.$$

Since $\widehat{\theta}$ is multiplicative, then hence we yield a ring homomorphism $\theta : A_{inf} \to \mathcal{O}_{\mathbb{C}_K}$ satisfying the proposition.

It remains to establish the surjectivity of $\theta$. Let $x$ be an arbitrary element in $\mathcal{O}_{\mathbb{C}_K}$. Since $\mathcal{O}_{\mathbb{C}_K}$ is $p$-adically complete, it is enough to find elements $c_0, c_1, \cdots \in \mathcal{O}_F$ with

$$x - \sum_{n=0}^{m} c_n^{\sharp} p^n \in p^{m+1}\mathcal{O}_{\mathbb{C}_K} \quad \text{for each } m = 0, 1, \cdots,$$

In fact, by lemma 3.11 we can inductively define each $c_m$ to be any element in $\mathcal{O}_F$ with

$$\frac{1}{p^m}\left(x - \sum_{n=0}^{m-1} c_n^{\sharp} p^n\right) - c_m^{\sharp} \in p\mathcal{O}_{\mathbb{C}_K},$$

thereby completing the proof. □

For the rest of this section, we let $\theta : A_{inf} \to \mathcal{O}_{\mathbb{C}_K}$ be the ring homomorphism constructed in Proposition 4.2, and let $\theta[1/p] : A_{inf}[1/p] \to \mathbb{C}_K$ be the induced map on $A_{inf}[1/p]$. We also choose an element $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$, and set $\xi := [p^\flat] - p \in A_{inf}$.

**Definition 4.3.** *We define the de Rham local ring by*

$$B_{dR}^{+}; + \varprojlim_{j} A_{inf}[1/p]/\ker(\theta[1/p])^j$$

*We denote by $\theta_{dR}^{+}$ the natural projection $B_{dR}^{+} \twoheadrightarrow A_{inf}[1/p]/\ker(\theta[1/p])$.*

**Goal:** Verfiy that $B_{dR}^{+}$ is a DVR. Define $B_{dR}$ as the fraction field of $B_{dR}^{+}$.

Recall de Rham cohomology admits a canonical filtration (Hodge filtration). also recall hodge-tate decomposition can be stated in terms of Hodge-Tate period ring $B_{HT}$. Hence construction $B_{dR}$ as a ring with canonical filtration which recovers $B_{HT}$ as associated graded algebra.

So construct subring $B_{dR}^{+}$ as a complete discrete valuation ring with an action of $\Gamma_K$ such that there exist $\Gamma_K$-equivariant isomorphisms

$$B_{dR}^{+}/\mathfrak{m}_{dR} \cong \mathbb{C}_K \quad \text{and} \quad \mathfrak{m}_{dR}/\mathfrak{m}_{dR}^2 \cong \mathbb{C}_K(1)$$

where $\mathfrak{m}_{dR}$ is the max ideal. In char $p$, look at $W(\mathcal{O}_F)$. Fontaine applied the Witt vector construction to the field $\mathbb{C}_K$ of characteristic 0 by passing to characteristic $p$. So what he does is, he defined $A_{inf}$ as ring of Witt vectors over the perfect ring

$$R_K := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K},$$

which he called the perfection of $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$, then constructed the homomorphism $\theta[1/p]$ as above to realize $\mathbb{C}_K$ as a quotient of $A_{inf}[1/p]$; indeed, since $R_K$ is naturally isomorphic to $\mathcal{O}_F$ by Proposition 3.7, our construction provides a modern interpretation for the construction of $R_K$ and $A_{inf}$. Fontaine then define $B_{dR}^{+}$ as the completion of $A_{inf}[1/p]$ with respect to $\ker(\theta[1/p])$ as in Definition 4.3, and showed that $B_{dR}^{+}$ satisfies all the desired properties. We now aim to show that $B_{dR}^{+}$ is a complete discrete valuation ring with $\mathbb{C}_K$ as the residue field. To this end we study several properties of $\ker(\theta)$.

We now aim to show that $B_{dR}^{+}$ is a complete discrete valuation ring with $\mathbb{C}_K$ as the residue field. To this end we study several properties of $\ker(\theta)$.

**Lemma 4.4.** *For each $n \geq 0$ we have* $\ker(\theta) \cap p^n A_{inf} = p^n \ker(\theta)$.

**Lemma 4.5.** *Every element $a \in \ker(\theta)$ is of the form $a = c\xi + dp$ for some $c, d \in A_{inf}$.*

**Proposition 4.6.** *The ideal* $\ker(\theta)$ *in $A_{inf}$ is generated by $\xi$.*

*Proof.* By definition we have

$$\theta(\xi) = \theta([p^\flat] - p) = (p^\flat)^\sharp - p = p - p = 0.$$

Hence we only need to show that $\ker(\theta)$ lies in the ideal $\xi A_{inf}$. Let a be an arbitrary element in $\ker(\theta)$. Since $A_{inf}$ is $p$-adically separated and complete by construction, it suffices to show that there exist elements $c_0, c_1, \cdots, \in A_{inf}$ with

$$a - \sum_{n=0}^{m} c_n \xi p^n \in p^{m+1} A_{inf} \quad \text{for each } m \geq 0.$$

We proceed by induction on $m$ to find such $c_0, c_1, \cdots \in A_{inf}$. As both $\xi$ and $a$ lie in $\ker(\theta)$, we have

$$a - \sum_{n=0}^{m-1} c_n \xi p^n \in \ker(\theta) \cap p^m A_{inf} = p^m \ker(\theta)$$

by the induction hypothesis and Lemma 4.4. Then by Lemma 4.5 we find some $c_m, d_m \in A_{inf}$ with

$$a - \sum_{n=0}^{m-1} c_n \xi p^n = p^m(c_m \xi + p d_m),$$

or equivalently

$$a - \sum_{n=0}^{m} c_n \xi p^n = p^{m+1} d_m$$

as desired. □

The above yields $A_{inf}/\xi A_{inf} \cong \mathcal{O}_{\mathbb{C}_K}$ as valuation rings.It shows $\xi A_{inf}$ allows us to recover $\mathbb{C}_K$ from its tilt $F$.

**Corollary 4.7.** *The ideal* $\ker(\theta[1/p])$ *in* $A_{inf}[1/p]$ *is generated by* $\xi$.

*Proof.* For every $a \in \ker(\theta[1/p])$, we have $p^n a \in \ker(\theta)$ for some $n > 0$. Hence the assertion follows from Proposition 4.6. □

In fact, our proof shows every generator of $\ker(\theta)$ generates $\ker(\theta[1/p])$.

**Lemma 4.8.** *Every* $a \in A_{inf}[1/p]$ *with* $\xi a \in A_{inf}$ *is an element in* $A_{inf}$.

**Lemma 4.9.** *For all* $j \geq 1$ *we have* $A_{inf} \cap \ker(\theta[1/p])^j = \ker(\theta)^j$.

*Proof.* We only need to show $A_{inf} \cap \ker(\theta[1/p])^j \subseteq \ker(\theta)^j$ since the reverse containment is obvious. Let $a$ be an arbitrary element in $A_{inf} \cap \ker(\theta[1/p])^j$. Corollary 4.7 implies that there exists some $r \in A_{inf}[1/p]$ with $a = \xi^j b$. Then we find $b \in A_{inf}$ by Lemma 4.8, and consequently obtain $a \in \ker(\theta)^j$ by Proposition 4.6. □

**Proposition 4.10.** *We have* $\bigcap_{j=1}^{\infty} \ker(\theta)^j = \bigcap_{j=1}^{\infty} \ker(\theta[1/p])^j = 0$.

*Proof.* By Lemma 4.9 we find that

$$\bigcap_{j=1}^{\infty} \ker(\theta[1/p])^j = \left( \bigcap_{j=1}^{\infty} \ker(\theta)^j \right) [1/p]$$

Hence it suffice to show that $\bigcap_{j=1}^{\infty} \ker(\theta)^j = 0$. Take an arbitrary element $c \in \bigcap_{j=1}^{\infty} \ker(\theta)^j$. As usual, let us write $c = \sum [c_n] p^n$ for some $c_n \in \mathcal{O}_F$. By proposition 4.6 we find that $c$ is divisible by arbitrarily high powers of $\xi = [p^\flat] - p$. This implies that $c_0$ is divisible by arbitrarily high powers of $p^\flat$, which in turn means $c_0 = 0$ as we have

$$\nu^\flat(p^\flat) = \nu((p^\flat)^\sharp) = \nu(p) = 1 > 0/$$

Hence we find some $c' \in A_{inf}$ with $c = pc'$. Moreover, Lemma 4.9 and the above together yield

$$c' \in A_{inf} \cap \left( \bigcap_{j=1}^{\infty} \ker(\theta)^j \right) [1/p] = A_{inf} \cap (\bigcap_{j=1}^{\infty} \ker(\theta[1/p])^j) = \bigcap_{j=1}^{\infty} \ker(\theta)^j.$$

Then an easy induction shows that $c$ is infinitely divisible by $p$, which in turn implies $c = 0$ as $A_{inf}$ is $p$-adically complete. □

**Corollary 4.11.** *The natural map*

$$A_{inf}[1/p] \to \varprojlim_j A_{inf}[1/p]/\ker(\theta[1/p])^j = B_{dR}^+$$

*is injective. In particular, we may canonically identify* $A_{inf}[1/p]$ *as a subring of* $B_{dR}^+$.

We also state the following theorem without proof:

**Proposition 4.12.** *The ring* $B_{dR}^+$ *is a complete discrete valuation ring with* $\ker(\theta_{dR}^+)$ *as the maximal ideal and CK as the residue field. Moreover, the element* $\xi$ *is a uniformizer of* $B_{dR}^+$.

The most important part to notice is that

$$B_{dR}^+/\ker(\theta_{dR}^+) \cong A_{inf}[1/p]/\ker(\theta[1/p]) \cong \mathbb{C}_K.$$

**Definition 4.13.** *We define the de Rham period ring* $B_{dR}$ *as the fraction field of* $B_{dR}^+$.

Our argument so far in this subsection remains valid if $\mathbb{C}_K$ is replaced by any algebraically closed perfectoid field of characteristic 0. Hence we may regard $B_{dR}$ as a functor from the category of algebraically closed perfectoid fields over $\mathbb{Q}_p$ to the category of complete valued fields.

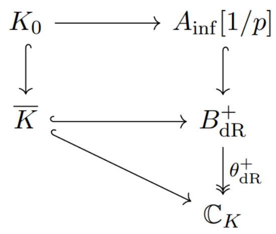
**Proposition 4.14.** *For every uniformizer* $\pi$ *of* $B_{dR}^+$*, the filtration* $\{\pi^n B_{dR}^+\}_{n\in\mathbb{Z}}$*of* $B_{dR}$ *satisfies the following properties:*

1. $\pi^{n+1}B_{dR}^+ \subseteq \pi^n B_{dR}^+$ *for all* $n \in \mathbb{Z}$
2. $\bigcap_{n\in\mathbb{Z}} \pi^n B_{dR}^+ = 0$ *and* $\bigcup_{n\in\mathbb{Z}} \pi^n B_{dR}^+ = B_{dR}$.
3. $(\pi^m B_{dR}^+)\cdot(\pi^n B_{dR}^+) \subseteq \pi^{m+n}B_{dR}^+$ *for all* $m, n \in \mathbb{Z}$.

**Remark.** The filtration $\{\pi^i B_{dR}^+\}_{n\in\mathbb{Z}}$ does not depend on the choice of $\pi$; indeed, we have an identification $\pi^i B_{dR}^+ = \ker(\theta_{dR}^+)^n$ for each $n \in \mathbb{Z}$.

**Proposition 4.15.** *Let* $W(k)$ *denote the ring of Witt vectors over* $k$*, and let* $K_0$ *denote the fraction field of* $W(k)$.

1. *The field* $K$ *is a finite totally ramified extension of* $K_0$.
2. *There exists a natural commutative diagram:*




*where the diagonal map is the natural inclusion.*

*Proof.* Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}_K$. The natural projection $\mathcal{O}_K/p\mathcal{O}_K \twoheadrightarrow \mathcal{O}_K/\mathfrak{m} = k$ admits a canonical section $s : k \to \mathcal{O}_K/p\mathcal{O}_K$; indeed, the ring $\mathcal{O}_K/p\mathcal{O}_K$ is a vector space over $k$ with basis given by $1, \pi, \cdots, \pi_{e-1}$, where $\pi$ is a uniformizer in $\mathcal{O}_K$ with $\nu(\pi) = 1/e$. In addition, the map $s$ induces a homomorphism of discretely valued fields $K_0 \to K$. We thus obtain the statement (1) by observing that both $K_0$ and $K$ are complete with the residue field k.

Let us now prove the statement (2). Since $k$ is perfect, the section $s : k \to \mathcal{O}_K/p\mathcal{O}_K$ induces a natural map

$$k \to \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} \cong \mathcal{O}_F$$

where the isomorphism is given by Proposition 3.7. We then obtain the top horizontal arrow in diagram above, and the upper right vertical arrow in the diagram above by Corollary 4.11. Hence $B_{dR}^+$ is a complete discrete valuation ring over $K_0$. Moreover, the statement (1) implies that $K$ is a separable algebraic extension of K0, thereby yielding the left vertical map in the diagram above. Now we deduce by Hensel's lemma that the subfield $K$ of the residue field $\mathbb{C}_K$ uniquely lifts to a subfield of $B_{dR}^+$ over $K_0$, thereby obtaining the middle horizontal arrow in above. □

**Final goal:** describe and study the natural action of $\Gamma_K$ on $B_{dR}$.

**Proposition 4.16.** *There exists a refinement of the discrete valuation topology on* $B_{dR}^+$ *that satisfies the following properties:*

1. *The natural map* $A_{inf} \to B_{dR}^+$ *identifies* $A_{inf}$ *as a closed subring of* $B_{dR}^+$.

2. *The map $\theta[1/p]$ is continuous and open with respect to the p-adic topology on $\mathbb{C}_K$.*
3. *There exists a continuous map $\log : \mathbb{Z}_p(1) \to B_{dR}^+$ with*

$$\log(\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\epsilon]-1)^n}{n} \quad \text{for every } \varepsilon \in \mathbb{Z}_p(1)$$

*under the natural identification $\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K}) = \{\varepsilon \in \mathcal{O}_F : \varepsilon^\sharp = 1\}$.*

4. *The multiplication by any uniformizer yields a closed embedding on $B_{dR}^+$.*
5. *The ring $B_{dR}^+$ is complete.*

Here we provide an indication on why Proposition 4.16 is necessary for our discussion. As we will soon describe, the natural $\Gamma_K$-action on $B_{dR}$ is induced by the action of $\Gamma_K$ on $\mathbb{C}_K$ such that the map $\theta_{dR}^+$ is $\Gamma_K$-equivariant. Proposition 4.16 ensures that the map $\theta_{dR}^+$ is furthermore continuous with respect to the $p$-adic topology on $\mathbb{C}_K$, thereby allowing us to exploit the topological properties of the $\Gamma_K$-action on $\mathbb{C}_K$.

For the rest of this chapter, we consider the map $\log : \mathbb{Z}_p(1) \to B_{dR}^+$ as given by Proposition 4.16. In addition, we fix a $\mathbb{Z}_p$-basis element $\varepsilon \in \mathbb{Z}_p(1)$ and write $t := \log(\varepsilon)$. We often regard $\varepsilon$ as an element in $\mathcal{O}_F$ via the identification $\mathbb{Z}_p(1) = \{c \in \mathcal{O}_F : c^\sharp = 1\}$ as noted in Proposition 4.16. We also regard $A_{inf}[1/p]$ as a subring of $B_{dR}^+$ in light of Corollary 4.11.

**Lemma 4.17.** *We have $v^\flat(\varepsilon - 1) = \frac{p}{p-1}$.*

*Proof.* By construction we may write $\varepsilon = (\xi_{p^n})$ where each $\xi_{p^n}$ is a primitive $p^n$-th root of unity in $K$. Then we find

$$\begin{aligned}
\nu^\flat(\varepsilon - 1) &= \nu\left((\varepsilon-1)^\sharp\right) = \nu\left(\lim_{n\to\infty}(\zeta_{p^n} - 1)^{p^n}\right) \\
&= \lim_{n\to\infty} p^n \nu(\zeta_{p^n} - 1) = \lim_{n\to\infty} \frac{p^n}{p^{n-1}(p-1)} \\
&= \frac{p}{p-1}
\end{aligned}$$

by the continuity of the valuation $\nu$.

□

**Lemma 4.18.** *The element $\xi$ divides $[\varepsilon] - 1$ in $A_{inf}$.*

*Proof.* By consturction we have

$$\theta([\varepsilon] - 1) = \varepsilon^\sharp - 1 = 1 - 1 = 0.$$

□

**Proposition 4.19.** *The element $t \in B_{dR}^+$ is a uniformizer.*

*Proof.* By Lemma 4.18 we have

$$[\varepsilon] - 1 \in \xi A_{inf} \quad \text{and} \quad t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \in \xi B_{dR}^+.$$

We also find $\frac{([\varepsilon]-1)^n}{n} \in \xi^2 B_{dR}^+$ for each $n \geq 2$. Since $\xi$ is a uniformizer of $B_{dR}^+$ as noted in Proposition 4.12, it suffices to prove $[\varepsilon] - 1 \notin \xi^2 B_{dR}^+$.

Suppose for contradiction that $[\varepsilon]-1$ lies in $\xi^2 B_{dR}^+$. Then the proof of Proposition 4.12 shows that the image of $[\varepsilon] - 1$ under the projection $B_{dR}^+ \in A_{inf}[1/p]/\ker(\theta[1/p])^2$ is zero. Since $[\varepsilon] - 1$ is an element of $A_{inf}$, we find $[\varepsilon] - 1 \in \ker(\theta[1/p])^2 \cap A_{inf}$. Hence Proposition 4.6 and Lemma 4.9 together imply that $[\varepsilon] - 1$ is divisible by $\xi^2$ in $A_{inf}$.

Since the first coefficients in the Teichmuller expansions for $[\varepsilon] - 1$ and $\xi^2$ are respectively equal to $[\varepsilon - 1]$ and $[(p^\flat)^2]$, we obtain

$$\nu^\flat(\varepsilon - 1) \geq \nu^\flat((p^\flat)^2) = 2\nu^\flat(p^\flat) = 2\nu((p^\flat)^\sharp) = 2\nu(p) = 2.$$

On the other hand, if $p$ is odd we have $\nu^\flat(\varepsilon - 1) < 2$ by Lemma 4.17. Therefore we find $p = 2$. Let us now take an element $c \in A_{inf}$ with $[\varepsilon] - 1 = \xi^2 c$. We then compare the coefficients of $p$ in the Teichmuller expansions of both sides and find $\varepsilon - 1 = c_1(p^\flat)^4$ where $c_1$ denote the coefficient of $p$ in the Teichmuller expansion of $c$. Hence we have

$$\nu^\flat(\varepsilon - 1) \geq \nu^\flat((p^\flat)^4) = 4\nu^\flat(p^\flat) = 4\nu((p^\flat)^\sharp) = 4\nu(p) = 4,$$

thereby obtaining a desired contradiction since Lemma 4.17 yields $\nu^\flat(\varepsilon - 1) = 2$. □

The proof above shows that the power series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n}$ converges with respect to te discete valuation topology on $B_{dR}^+$.

**Lemma 4.20.** *For every $m \in \mathbb{Z}_p$ we have $\log(\varepsilon^m) = m\log(\varepsilon)$.*

**Theorem 4.21** (Fontaine(Fon82))**.** *The natural action of $\Gamma_K$ on $B_{dR}$ satisfies the following properties:*

1. *The logarithm map and $\theta_{dR}^+$ are $\Gamma_K$-equivariant.*
2. *For every $\gamma \in \Gamma_K$, we have $\gamma(t) = \chi(\gamma)t$.*
3. *Each $t^n B_{dR}^+$ is stable under the action of $\Gamma_K$.*
4. *There exists a canonical $\Gamma_K$-equivariant isomorphism*

$$\bigoplus_{n\in\mathbb{Z}} t^n B_{dR}^+/t^{n+1}B_{dR} \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_K(n) = B_{HT}.$$

5. *$B_{dR}$ is $(\mathbb{Q}_p, \Gamma_K)$-regular with a natural identification $B_{dR}^{\Gamma_K} \cong K$.*

*Proof.* Let us first describe the natural action of $\Gamma_K$ on $B_{dR}$. The action of $\Gamma_K$ on $\mathbb{C}_K$ naturally induces an action on $F = \varprojlim_{x\mapsto x^p} \mathbb{C}_K$ as the $p$-th power map on $\mathbb{C}_K$ is $\Gamma_K$-equivariant.

More precisely, given an arbitrary element $x = (x_n) \in F$ we have $\gamma(x) = (\gamma(x_n))$ for every $\gamma \in \Gamma_K$. It is then evident that $\mathcal{O}_F$ is stable under the action of $\Gamma_K$. Hence by functoriality of Witt vectors we obtain a natural action of $\Gamma_K$ on $A_{inf}[1/p]$ with

$$\Gamma(\sum [c_n]p^n) = \sum [\gamma(c_n)]p^n \qquad \text{for all } \gamma \in \Gamma_K, c_n \in \mathcal{O}_F.$$

We then find that $\theta$ and $\theta[1/p]$ are both $\Gamma_K$-equivariant by construction, and consequently deduce that both $\ker(\theta)$ and $\ker(\theta[1/p])$ are stable under the action of $\Gamma_K$. Therefore $\Gamma_K$ naturally acts on $B_{dR}^+ = \varprojlim_j A_{inf}[1/p]/\ker(\theta[1/p])^j$ and its fraction field $B_{dR}$.

With our discussion in the preceding paragraph, it is straightforward to verify the property (i). Moreover, for every $\gamma \in \Gamma_K$ we use Lemma 4.20 to find

$$\gamma(t) = \gamma(\log(\varepsilon)) = \log(\gamma(\varepsilon)) = \log(\varepsilon^{\chi(\gamma)}) = \chi(\gamma)\log(\varepsilon) = \chi(\gamma)t,$$

thereby deducing the property (ii). The property (iii) then immediately follows as $B^+dR$ is stable under the action of $\Gamma_K$.

Let us now prove the property (iv). We note that the natural isomorphism

$$B_{dR}^+/\ker(\theta_{dR}^+) = B_{dR}^+/tB_{dR}^+ \cong A_{inf}[1/p]/\ker(\theta[1/p]) \cong \mathbb{C}_K.$$

is $\Gamma_K$-equivariant, and consequently obtain $\Gamma_K$-equivariant isomorphisms

$$\ker(\theta_{dR}^+)^n/\ker(\theta_{dR}^+)^{n+1} = t^n B_{dR}^+/t^{n+1}B_{dR}^+ \cong \mathbb{C}_K(n)$$

for all $n \in \mathbb{Z}$ by the property (ii)
These isomorphisms are canonical since $t$ is uniquely determined up to $\mathbb{Z}_p^\times$-multiple by Lemma 4.20. We thus obtain the desired $\Gamma_K$-equivariant isomorphism by taking the direct sum of the above isomorphisms.

It remains to verify the property (v). The field $B_{dR}$ is $(\mathbb{Q}_p, \Gamma_K)$-regular. In addition, since the map $\theta_{dR}^+$ is $\Gamma_K$-equivariant by construction, the natural injective homomorphism $\overline{K} \to B_{dR}^+$ given by Proposition 4.15 is also $\Gamma_K$-equivariant, thereby inducing an injective homomorphism

$$K = \overline{K}^{\Gamma_K} \hookrightarrow (B_{dR}^+)^{\Gamma_K} \hookrightarrow B_{dR}^{\Gamma_K}. \tag{4}$$

Then by properties (iii) and (iv) we get an injective $K$-algebra homomorphism

$$\bigoplus_{n \in \mathbb{Z}} (B_{dR}^{\Gamma_K} \cap t^n B_{dR}^+)/(B_{dR}^{\Gamma_K} \cap t^{n+1} B_{dR}^+) \hookrightarrow B_{HT}^{\Gamma_K}$$

Since we have $B_{HT}^{\Gamma_K} \cong K$, the $K$-algebra on the source has dimension at most 1. Hence we find $\dim_K B_{dR}^{\Gamma_K} \leq 1$, thereby completing the proof by (4). □