p-adic Hodge Theory (Spring 2023): Week 8

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This week: Generic Fibers of p-divisble groups

1 Generic Fibres of p-divisible groups

The main focus of this subsection is to prove the second main result for this chapter, which says that the generic fiber functor on the category of p-divisible groups over $\mathcal{O}_K$ is fully faithful.

Theorem 1.1 (Tate)**.** *The generic fiber functor for the category of p-divisible groups over $\mathcal{O}_K$ is fully faithful.*

We first note the following result:

Proposition 1.2. *Let $G = \varinjlim G_v$ be a p-divisible group of height h and dimension d over $\mathcal{O}_K$. Let $G_v = \mathrm{Spec}(A_v)$ where A_v is a finite free $\mathcal{O}_K$-algebra. Then the discriminant ideal of A_v over $\mathcal{O}_K$ is generated by $p^{dvp^{hv}}$.*

Sketch of Proof. Recall that we have an exact sequence

$$0 \to G_1 \xrightarrow{i_{v,1}} G_{v+1} \xrightarrow{j_{1,v}} G_v \to 0$$

We can then show that

$$\mathrm{disc}(A_{v+1}/\mathcal{O}_K) = \mathrm{disc}(A_v/\mathcal{O}_K)^{p^h} \cdot \mathrm{disc}(A_1)^{p^{hv}}.$$

By induction, we reduce to the case $v = 1$. The connected–étale sequence is

$$0 \to G_1^\circ \to G_1 \to G_1^{\text{ét}} \to 0.$$

We can show that $\mathrm{disc}(A^{\text{ét}}/\mathcal{O}_K) = (1)$. It is hence enough to show that $\mathrm{disc}(A_1^\circ/\mathcal{O}_K) = (p^{d \cdot p^h})$. Using Serre–Tate correspondence

$$A_1 = \mathcal{O}_K \otimes_{A_1[p]_\mu} A$$

and

$$\mathrm{disc}(A_1/\mathcal{O}_K) = \mathrm{disc}(\mathcal{A}/[p]\mathcal{A}).$$

For proof of this, check Haines' notes.

□

Lemma 1.3. *Consider a homomorphism $f : G \to H$ between p-divisible groups. If $\widetilde{f} : G \times \mathcal{O}_K K \to H \times_{\mathcal{O}_K} K$ is an isomorphism, f is an isomorphism*

Proof. Let $G = \varinjlim_v G_v$, $H = \varinjlim_v H_v$, $G_v = \mathrm{Spec}(A_v)$, $H_v = \mathrm{Spec}(B_v)$. The map f consists of maps $\alpha_v : B_v \to A_v$ such that $\alpha_v \otimes 1 : B_v \otimes K \xrightarrow{\cong} A_v \otimes K$.
Since both A_v, B_v are finite free over $\mathcal{O}_K$, $B_v \hookleftarrow A_v$. If $\mathrm{disc}(A_v/\mathcal{O}_K) = \mathrm{disc}(B_v/\mathcal{O}_K)$, then we are done. Recall that $\dim(G)$ is determined by $T_p(G)$. □

Remark 1.4. *This statement is not true for finite flat $\mathcal{O}_K$-group schemes. However, if $K/\mathbb{Q}_p$ is finite with $e < p - 1$, then Lemma 1.3 also holds (this is a Theorem of Raynaud).*

Proposition 1.5. *Let G be a p-divisible group over $\mathcal{O}_K$, and let M a $\mathbb{Z}_p$-direct summand of $T_p(G)$ which is stable under the action of Γ_K. There exists a p-divisible group H over $\mathcal{O}_K$ with a homomorphism $\iota : H \to G$ which induces an isomorphism $T_p(H) \cong M$.*

Proof. There is a p-divisible group $\widetilde{H}$ over K with $\widetilde{H} \to G \times_{\mathcal{O}_K} K$ such that $T_p(\widetilde{H}) \cong H$, where $\widetilde{H} = \varinjlim \widetilde{H}_v$ $\square$

Consider the scheme closure $\underline{H_v}$ of $\widetilde{H}_v$ in G_v.

Remark 1.6. *The injective limit $\varinjlim_v H_v$ may not be a p-divisible group over $\mathcal{O}_K$.*

We get maps $\underline{H_v} \hookrightarrow \underline{H_{v+1}}$ induced from $\widetilde{H}_v \hookrightarrow \widetilde{H}_{v+1}$.

We claim that there exists v_0 such that

$$H_v = \underline{H_{v+v_0}}/\underline{H_{v_0}}$$

such that $\varinjlim H_v$ is a p-divisible group.
On the generic fiber,

$$H_v \times K \cong \widetilde{H}_{v+v_0}/\widetilde{H}_{v_0} \cong \widetilde{H}_v.$$

The map $[p]$ on H_{v+1} factors through H_v, since $\widetilde{H}_{v+1}/\widetilde{H}_v$ is killed by p, so $\underline{H_{v+1}}/\underline{H_v}$ is killed by p. Hence $[p]$ induces:

$$\delta_v : \underline{H_{v+2}}/\underline{H_{v+1}} \to \underline{H_{v+1}}/\underline{H_v}$$

On generic fibers, δ_v is an isomorphism. Writing $\underline{H_{v+1}}/\underline{H_v} = \mathrm{Spec}(B_v)$, δ_v induces a map

$$B_v \to B_{v+1}$$

which becomes an isomorphism after tensoring with K. Hence $B_v \hookrightarrow B_{v+1}$ and $\{B_v\}$ is an increasing order in $B_1 \otimes K$.
Fact. The integral closure of $\mathcal{O}_K$ in $B_1 \otimes K$ is Noetherian. Hence there exists v_0 such that

$$B_v \cong B_{v+1} \text{ for all } v \geq v_0.$$

If $v \geq v_0$, we have that

$$\underline{H_{v+2}}/\underline{H_{v+1}} \cong \underline{H_{v+1}}/\underline{H_v}.$$

Now,

$$\begin{array}{ccccc} H_{v+1} = \underline{H_{v+1+v_0}}/\underline{H_{v_0}} & \xrightarrow{[p^v]} & \underline{H_{v+1+v_0}}/\underline{H_{v_0}} \cong H_{v+1} \\ \downarrow\!\!\downarrow & & \uparrow \\ \underline{H_{v+1+v_0}}/\underline{H_{v_0+v}} & \xrightarrow{\cong} & \underline{H_{v_0+1}}/\underline{H_{v_0}} \end{array}$$

Finally, $\ker([p^v]) = \underline{H_{v+v_0}}/\underline{H_{v_0}} = H_v$.

Proposition 1.7. *There is a bijection:*

$$\mathrm{Hom}(G, H) \cong \mathrm{Hom}(G \times_{\mathcal{O}_K} K, H \times_{\mathcal{O}_K} K).$$

Proof. If you have a homomorphism $f : G \times K \to H \times K$. Then $\widetilde{f}$ uniquely extends to $f : G \to H$.

For uniqueness: if $G_v = \mathrm{Spec}(A_v)$, $H_v = \mathrm{Spec}(B_v)$, then $\widetilde{f}_v : B_v \otimes K \to A_v \otimes K$, so there is at most one extension to $B_v \to A_v$ (by choosing generators).
We need to show existence. Consider the graph of $T = T_p f : T_p(G) \to T_p(H)$:

$$M \subseteq T_p(G) \oplus T_p(H).$$

We claim that M is a $\mathbb{Z}_p$-direct summand. Note that

$$T_p(G) \oplus T_p(H)/M \xrightarrow{\cong} T_p(H)$$

$$(x, y) \mapsto y - T(x),$$

so $T_p(G) \oplus T_p(H)/M$ is torsion-free. Hence the short exact sequence

$$0 \to M \to T_p(G) \oplus T_p(H) \to T_p(G) \oplus T_p(H)/M \to 0$$

splits.
Since $T_p(G \times H) = T_p(G) \oplus T_p(H)$, by Proposition 1.5, there exists a p-divisible group G' over $\mathcal{O}_K$ with a homomorphism $\iota : G' \to G \times H$ such that $T_p(G') \cong M$.
Consider the projection maps

$$\pi_1 : G \times H \to G.$$

$$\pi_2 : G \times H \to H.$$

Then $\pi_1 \circ \iota : G' \to G$ is an isomorphism by lemma 1.3. Then $f = \pi_2 \circ \iota \circ (\pi_1 \circ \iota)^{-1}$ extends $\widetilde{f}$. □

Remark 1.8. *As a related fact, the special fiber functor on the category of p-divisible groups over $\mathcal{O}_K$ is faithful. In other words, for arbitrary p-divisible groups G and H over $\mathcal{O}_K$, the natural map*

$$\hom(G, H) \to \mathrm{Hom}(G \times_{\mathcal{O}_K} k, H \times_{\mathcal{O}_K} k)$$

is injective. A complete proof of this fact can be found in [CCO14, Proposition 1.4.2.3].

It is also worthwhile to mention that Proposition 1.7 remains true if the base ring $\mathcal{O}_K$ is replaced by any ring R that satisfies the following properties:

1. R is integrally closed and noetherian,
2. R is an integral domain whose fraction field has characteristic 0.

In fact, it is not hard to deduce the general case from Theorem 1.7 by algebraic Hartog's Lemma.

Corollary 1.9. *For arbitrary p-divisible groups G and H over OK, the natural map*

$$\mathrm{Hom}(G, H) \to \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G), T_p(H))$$

is bijective.

We conclude this section by stating a fundamental theorem which provides a classification of p-divisible groups over $\mathcal{O}_K$ when K is unramified over $\mathbb{Q}_p$. We write $W(k)$ for the ring of Witt vectors over k.

Definition 1.10. *A Honda system over $W(k)$ is a Dieudonné module M over k together with a $W(k)$-submodule L such that φM induces an isomorphism $L/pL \cong M/\varphi_M(M)$.*

Theorem 1.11. *If $p > 2$, there exists an anti-equivalence of categories*

$$\{ \text{ p-divisible groups over W(k) } \} \xrightarrow{\cong} \{ \text{ Honda systems over W(k) } \}$$

such that for every p-divisible group G over $W(k)$ with the mod p reduction $G := G \times_{W(k)} k$, the Dieudonné module of the associated Honda system coincides with $\mathbb{D}(\overline{G})$.

2 Period rings and functors

The goal is to define and study:

- period rings B_{HT}, B_{dR}, B_{cris}.
- de Rham and crystalline representations.

There is another important period ring, B_{st}, related to semistable representations. We will omit this here entirely.

2.1 Fontaine's formalism on Period rings

The reference for this section is [BC09, Section 5]. Let K be a p-adic field and Γ_K be the absolute Galois group $Gal(\overline{K}/K)$ and $I_K = Gal(\overline{K}/K^{un})$ be the inertia group of K.

Definition 2.1. *Let B be a $\mathbb{Q}_p$-algebra with an action of Γ_K and let C be the fraction field of B with the natural Γ_K-action. We say that B is $(\mathbb{Q}_p, \Gamma_K)$-regular if*

- $B^{\Gamma_K} = C^{\Gamma_K}$
- *any $b \in B$ with $b \neq 0$ is a unit if $\mathbb{Q}_p \cdot b$ is stable under the Γ_K-action.*

Example 2.2. *Every field extension of $\mathbb{Q}_p$ under any Γ_K-action is $(\mathbb{Q}_p, \Gamma_K)$-regular.*

Remark 2.3. *If F is a field and G is a group, we can define (F, G)-regular rings by replacing $\mathbb{Q}_p$ with F and Γ_K with G in the above definition.*

We can also extend our formalism to this setting.

Definition 2.4. *Suppose B is a $(\mathbb{Q}_p, \Gamma_K)$-regular ring and $E = B^{\Gamma_K}$. Then*

1. *for all $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, define*
$$D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}.$$
2. *a representation $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-admissible if*
$$\dim_E D_B(V) = \dim_{\mathbb{Q}_p} V.$$

We denote by $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ the category of B-admissible p-adic representations.

Remark 2.5. *Let R be a topological ring with a continuous Γ_K-action, then*

$$H^1(\Gamma_K, GL_d(R)) = \{ \text{ continuous } d\text{-dimensional semilinear } \Gamma_K \text{ representations over } R\}/\cong .$$

For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, we can consider the class $[V] \in H^1(\Gamma_K, GL_n(\mathbb{Q}_p))$. Let $[V]_B$ be its image in $H^1(\Gamma_K, GL_n(B))$. Then V is B-admissible if and only if $[V]_B$ is trivial.

Example 2.6.
1. *For any $(\mathbb{Q}_p, \Gamma_K)$-regular $B, V = \mathbb{Q}_p$ with trivial Γ_K-action is B-admissible. Indeed, $D_B(V) = B^{\Gamma_K} = E$.*
2. *Consider $B = \overline{K}$. Then $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is K-admissible if and only if V is potentially trivial (i.e. the action of Γ_K on V factors through some finite quotient). This follows from the group cohomology interpretation and Hilbert 90.*
3. *Consider $B = \mathbb{C}_K$. Then $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is $\mathbb{C}_K$-admissible if and only if V is potentially unramified, i.e. the action of the inertia group factors through a finite quotient. This fact is quite difficult; it follows from Sen theory and is almost as difficult as the Tate–Sen theorem.*

Definition 2.7. *Let $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ be a character. For every $\mathbb{Q}_p[\Gamma_K]$-module M, we define its twist by η to be the $\mathbb{Q}_p[\Gamma_K]$-module*
$$M(\eta) := M \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)$$
where $\mathbb{Q}_p(\eta)$ denotes the Γ_K-representation on $\mathbb{Q}_p$ given by η.

Example 2.8. *Given a $\mathbb{Q}_p[\Gamma_K]$-module M, we have an identification $M(n) \cong M(\chi^n)$ for every $n \in \mathbb{Z}$*

Lemma 2.9. *The group $\chi(I_K)$ is infinite.*

Proof. By definition χ encodes the action of Γ_K on $\mu_{p^\infty}(K)$. In particular, we have $\ker(\chi) = \mathrm{Gal}(K(\mu_{p^\infty}(\overline{K}))/K)$. Hence it suffices to show that $K(\mu_{p^\infty}(K))$ is infinitely ramified over K.
Let e_n be the ramification degree of $K(\mu_{p^n}(K))$ over K, and let e be the ramification degree of K over $\mathbb{Q}_p$. Then $e_n \cdot e$ is greater than or equal to the ramification degree of $\mathbb{Q}_p(\mu_{p^{n-1}}(K))$ over $\mathbb{Q}_p$, which is equal to $p^{n-1}(p-1)$. We thus find that en grows arbitrarily large as n goes to ∞, thereby deducing the desired assertion. □

Theorem 2.10. *Let $\eta : \Gamma_K \to \mathbb{Z}_p^\times$ be a continuous character. Then for $i = 0, 1$ we have canonical isomorphisms*
$$H^i(\Gamma_K, \mathbb{C}_K(\eta)) \cong \begin{cases} K & \text{if } \eta(I_K) \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.11. *Theorem 1.1.8 recovers the essential part of the Tate-Sen theorem: indeed, if we take $\eta = \chi^n$ for some $n \in \mathbb{Z}$, then Theorem 2.10 yields canonical isomorphisms*

$$H^0(\Gamma_K, \mathbb{C}_K(n)) \cong H^1(\Gamma_K, \mathbb{C}_K(n)) \cong \begin{cases} K & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$

by Example 2.8 and Lemma 2.9. Moreover, for $i = 0$ Theorem 2.10 says that $\mathbb{Q}_p(\eta)$ is $\mathbb{C}_K$-admissible if and only if it is potentially unramified, as we have already mentioned in Example 2.6.

Definition 2.12. *We define the Hodge-Tate period ring by*

$$B_{HT} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_K(n).$$

Proposition 2.13. *The Hodge-Tate period ring B_{HT} is $(\mathbb{Q}_p, \Gamma_K)$-regular.*

Proposition 2.14. *A p-adic representation V of Γ_K is Hodge-Tate if and only if it is B_{HT}-admissible.*

Proof. By definition we have

$$D_{B_{HT}}(V) = (V \otimes_{\mathbb{Q}_p} B_{HT})^{\Gamma_K} = \bigoplus_{n\in\mathbb{Z}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K}.$$

Define $\widetilde{\alpha}_V$ as in the theorem of Serre-Tate. Since $\widetilde{\alpha}_V$ is injective, it is an isomorphism if and only if the source and the target have the same dimension over $\mathbb{C}_K$, which amounts to the identity $\dim_K D_{B_{HT}}(V) = \dim_{\mathbb{Q}_p} V$. The desired assertion now follows from definition of Hodge-Tate representations and B_{HT}-admissibility. □

Example 2.15. *Let V be a p-adic representation of Γ_K which fits into an exact sequence*

$$0 \to \mathbb{Q}_p(\ell) \to V \to \mathbb{Q}_p(m) \to 0$$

where ℓ and m are distinct integers. We assert that V is Hodge-Tate. For every $n \in \mathbb{Z}$ we obtain an exact sequence

$$0 \to \mathbb{C}_K(\ell + n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n) \to \mathbb{C}_K(m + n) \to 0$$

as $\mathbb{C}_K(n)$ is flat over $\mathbb{Q}_p$, and consequently get a long exact sequence

$$0 \to \mathbb{C}_K(l + n)^{\Gamma_K} \to (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \to \mathbb{C}_K(m + n)^{\Gamma_K} \to H^1(\Gamma_K, \mathbb{C}_K(m + n))$$

Then from Hodge-Tate theorem in chapter 2 we have

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \cong \begin{cases} K & \text{for } n = -\ell, -m \\ 0 & \text{for } n \neq -\ell, -m \end{cases}$$

Hence we have

$$\dim_K D_{B_{HT}}(V) = \sum_{n\in\mathbb{Z}} \dim_K (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} = 2 = \dim_{\mathbb{Q}_p} V$$

thereby deducing the desired assertion.

Note a self extension of $\mathbb{Q}_p$ may not be Hodge-Tate.

Proposition 2.16. *Let $\eta : \Gamma_K \to \mathbb{Z}_p^\times$ be a continuous character. Then $\mathbb{Q}_p(\eta)$ is HodgeTate if and only if there exists some $n \in \mathbb{Z}$ such that $(\eta\chi^n)(I_K)$ is finite.*

Definition 2.17. *Let V be a Hodge-Tate representation. We say that an integer $n \in \mathbb{Z}$ is a Hodge-Tate weight of V with multiplicity m if we have*

$$\dim_K (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} = m > 0.$$

Example 2.18. *We record the Hodge-Tate weights for some Hodge-Tate representations.*

1. *For every $n \in \mathbb{Z}$ the Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is a Hodge-Tate representation with the Hodge-Tate weight $-n$.*

2. *For every p-divisible group G over $\mathcal{O}_K$, the rational Tate module $V_p(G)$ is a HodgeTate representation with the Hodge-Tate weights 0 and -1.*

3. *For an abelian variety A over K with good reduction, the ´etale cohomology $H^n_{ét}(A_{\overline{K}}, \mathbb{Q}_p)$ is a Hodge-Tate representation with the Hodge-Tate weights $0, 1, \cdots, n$.*