p-adic Hodge Theory (Spring 2023): Week 7

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This week: Proof of Hodge-Tate Decomposition
In this subsection, we derive the first main result for this chapter by exploiting our accumulated knowledge of finite flat group schemes and p-divisible groups. Let us first present some easy but useful lemmas.

1 Proof of Hodge-Tate Decomposition

Lemma 1.1. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. For each v we have canonical isomorphisms*

$$G_v(\overline{K}) \cong G_v(\mathbb{C}_K) \cong G_v(\mathcal{O}_{\mathbb{C}_K}).$$

Proof. Since $\mathbb{C}_K$ is algebraically closed, the first isomorphism follows from the fact that the generic fiber of G_v is étale. The second isomorphism is a direct consequence of the valuative criterion. □

Lemma 1.2. *For every p-divisible group G over $\mathcal{O}_K$ we have*

$$G(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} = G(\mathcal{O}_K) \quad \text{and} \quad t_G(\mathbb{C}_K)^{\Gamma_K} = t_G(K),$$

Proof. It should follow immediately, recalling that $\mathbb{C}_K^{\Gamma_K} = K$ and $\mathcal{O}_{\mathbb{C}_K}^{\Gamma_K} = \mathcal{O}_K$. □

Lemma 1.3. *Given a p-divisible group G over $\mathcal{O}_K$ we have*

$$\bigcap_{n=1}^{\infty} p^n G^\circ(\mathcal{O}_K) = 0.$$

Proof. As the valuation on K is discrete, there exists a minimum positive valuation δ; indeed, we have $\delta = \nu(\pi)$ where π is a uniformizer of K. Then an easy induction using Lemma from previous lecture yields $p^n G^\circ(\mathcal{O}_K) \subseteq \mathrm{Fil}^{n\delta} G^\circ(\mathcal{O}_K)$ for all $n \geq 1$. We thus deduce the desired assertion by observing $\cap_{n=1}^{\infty} \mathrm{Fil}^{n\delta} G^\circ(\mathcal{O}_K) = 0$. □

The main technical ingredient for this subsection is the interplay between the Tate modules and Cartier duality.

Definition 1.4. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. We define the Tate module of G by*

$$T_p(G) := T_p(G \times_{\mathcal{O}_K} K) = \varprojlim G_v(\overline{K}),$$

and the Tate comodule G by

$$\Phi_p(G) ;= \varinjlim G_v(\overline{K}).$$

Remark 1.5. *The Tate comodule $\Phi_p(G)$ is nothing other than $G(K)$, where G is regarded as a fpqc sheaf.*

Example 1.6. *$T_p(\mu_{p^\infty}) = \mathbb{Z}_p(1)$, and that $\Phi(\mu_{p^\infty}) = \varinjlim_{\mu_{p^v}}(\overline{K}) = \mu_{p^\infty}(\overline{K})$ is the group of p-power roots of unity in $\overline{K}$.*

Proposition 1.7. *Given a p-divisible group G over $\mathcal{O}_K$, Cartier duality induces natural Γ_K-equivariant isomorphisms*

$$T_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)) \quad \text{and} \quad \phi_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mu_{p^\infty}(\overline{K}))$$

ideas of proof. Note that every finite flat group scheme over K is étale. For each v we have a natural identification

$$G_v(\overline{K}) \cong (G_v^\vee)^\vee(\overline{K}) = \mathrm{Hom}_{\overline{K}\text{-grp}}((G_v^\vee)_{\overline{K}}, (\mu_{p^v})_{\overline{K}}) \cong \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^v}(\overline{K}))$$

Then the result is more of plug in of definition.

$$\begin{aligned}\Phi_p(G) &= \varprojlim G_v(\overline{K}) \\ &= \varprojlim_{\overline{K}}((G_v^\vee)_{\overline{K}}, (\mu_{p^v})_{\overline{K}}) \\ &= \varprojlim(G_v^\vee(\overline{K}), \mu_{p^\infty}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(\varprojlim G_v^\vee(\overline{K}), \mu_{p^\infty}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mu_{p^\infty}(\overline{K})),\end{aligned}$$

□

Proposition 1.8. *We have a short exact sequence*

$$0 \to \Phi_p(G) \to G(\mathcal{O}_{\mathbb{C}_K}) \xrightarrow{\log_G} t_G(\mathbb{C}_K) \to 0.$$

Proof. We know that $\Phi_p(G) = G(K) \subseteq G(\mathcal{O}_{\mathbb{C}_K})$. We need to check that $\log_G$ is surjective and its kernel is $\Phi_p(G)$. Recall that $\log_G$ induces an isomorphism $G(\mathcal{O}_{\mathbb{C}_K}) \otimes \mathbb{Q}_p \cong t_G(\mathbb{C}_K)$, so $\log_G$ is surjective after inverting p. Since $\mathbb{C}_K$ is algebraically closed, $G(\mathcal{O}_{\mathbb{C}_K})$ is p-divisible (i.e. multiplication by p on $G(\mathcal{O}_{\mathbb{C}_K})$ is surjective). Hence p is already invertible in $G(\mathcal{O}_{\mathbb{C}_K})$, showing that $\log_G$ is surjective. We now want to show that $\ker(\log_G) = \Phi_p(G)$. Then

$$\begin{aligned}\ker(\log_G) &= G(\mathcal{O}_{\mathbb{C}_K})_{\mathrm{tors}} \\ &= \varinjlim_v \varprojlim_i G_v(\mathcal{O}_{\mathbb{C}_K}/\mathfrak{m}^i\mathcal{O}_{\mathbb{C}_K}) \\ &= \varinjlim_v G_v(\mathcal{O}_{\mathbb{C}_K}) \\ &= \varinjlim_v G_v(\overline{K}) \\ &= \Phi_p(G),\end{aligned}$$

as to be shown. □

Example 1.9. *Let $G = \mu_{p^\infty}$. Then*

$$0 \to \mu_{p^\infty}(\overline{K}) \to 1 + \mathfrak{m}_{\mathbb{C}_K} \xrightarrow{\log_{\mu_{p^\infty}}} \mathbb{C}_K \to 0.$$

Proposition 1.10. *Every p-divisible group G over $\mathcal{O}_K$ gives rise to a commutative diagram of exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_p(G) & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\log_G} & t_G(\mathbb{C}_K) & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow \alpha & & \downarrow d\alpha & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mu_{p^\infty}(\overline{K})) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & 0 \end{array}$$

where α and $d\alpha$ are Γ_K-equivariant and injective.

Proof. The top row is as described in Proposition 1.8. The bottom row is induced by the short exact sequence in Example 1.9, and is exact since $T_p(G^\vee)$ is free over $\mathbb{Z}_p$. The left vertical arrow is the natural Γ_K-equivariant isomorphism given by Proposition 1.7.

Let us now construct the maps α and $d\alpha$. As usual, we write $G = \varinjlim G_v$ where G_v is a finite flat $\mathcal{O}_K$-group scheme. Hence we have

$$\begin{aligned}T_p(G^\vee) &= \varprojlim G_v^\vee(\overline{K}) \cong \varprojlim G_v^\vee(\mathcal{O}_{\mathbb{C}_K}) \\ &= \varprojlim \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_K}\text{-grp}}\left((G_v)_{\mathcal{O}_{\mathbb{C}_K}}, (\mu_{p^v})_{\mathcal{O}_{\mathbb{C}_K}}\right) \\ &= \mathrm{Hom}_{p\text{-div grp}}(G \times_{\mathcal{O}_K} \mathcal{O}_{\mathbb{C}_K}, (\mu_{p^\infty})_{\mathcal{O}_K}).\end{aligned} \tag{3.7}$$

We define map $\alpha : G(\mathcal{O}_{\mathbb{C}_k}) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K})$ by setting

$$\alpha(g)(u) := u_{\mathcal{O}_{\mathbb{C}_K}}(g) \quad \text{for each } g \in G(\mathcal{O}_{\mathbb{C}_K}) \text{ and } u \in T_p(G^\vee),$$

where $u_{\mathcal{O}_{\mathbb{C}_K}} : G(\mathcal{O}_{\mathbb{C}_k}) \to \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) \cong 1+\mathfrak{m}_{\mathbb{C}_K}$ is the map induced by u under the identification 3.7 above. We also define the map $d\alpha : t_G(\mathbb{C}_K) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$ by setting

$$d\alpha(z)(u) := du_{\mathbb{C}_K}(z) \quad \text{for each } z \in t_G(\mathbb{C}_K) \text{ and } u \in T_p(G^\vee),$$

where $du_{\mathbb{C}_K} : t_G(\mathbb{C}_K) \to t_{\mu_{p^\infty}}(\mathbb{C}_K) \cong \mathbb{C}_K$ is the map induced by u under the identification 3.7.
The maps α and $d\alpha$ are evidently $\mathbb{Z}_p$-linear and Γ_K-equivariant by construction. The commutativity of the left square follows by observing that the left vertical arrow can be also defined as the restriction of α on $G(\mathcal{O}_{\mathbb{C}_K}) \cong \Phi_p(G)$. The commutativity of the right square amounts to the commutativity of the following diagram

$$\begin{array}{ccc} G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\log_G} & t_G(\mathbb{C}_K) \\ \downarrow & & \downarrow \\ \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) = 1+\mathfrak{m}_{\mathbb{C}_K} & \xrightarrow{\log_{\mu_{p^\infty}}} & t_{\mu_{p^\infty}} = \mathbb{C}_K \end{array}$$

which is straightforward to verify by definition; indeed, the logarithm map yields a natural transformation between the functor of $\mathcal{O}_{\mathbb{C}_K}$-valued formal points and the functor of tangent space with values in K.

It remains to prove that α and $d\alpha$ are injective. By snake lemma we have $\mathbb{Z}_p$-linear isomorphisms

$$\ker(\alpha) \cong \ker(d\alpha) \quad \mathrm{coker}(\alpha) \cong \mathrm{coker}(d\alpha) \tag{3.8}$$

Hence it suffices to show that $d\alpha$ is injective.
As both $t_G(\mathbb{C}_K)$ and $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G_\vee), \mathbb{C}_K)$ are $\mathbb{Q}_p$-vector spaces, the $\mathbb{Z}_p$-linear map $d\alpha$ is indeed $\mathbb{Q}_p$-linear. Therefore both $\ker(d\alpha)$ and $\mathrm{coker}(d\alpha)$ are $\mathbb{Q}_p$-vector spaces. The isomorphisms (3.8) then tells us that both $\ker(\alpha)$ and $\mathrm{coker}(\alpha)$ are $\mathbb{Q}_p$-vector spaces as well. We assert that α is injective on $G(\mathcal{O}_K)$. Suppose for contradiction that $\ker(\alpha)$ contains a nonzero element $g \in G(\mathcal{O}_K)$. As $\ker(\alpha)$ is torsion free for being a $\mathbb{Q}_p$-vector space, we may assume $g \in G^\circ(\mathcal{O}_K)$. Let us define the map

$$\alpha^\circ : G^\circ(\mathcal{O}_{\mathbb{C}_K}) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\circ)^\vee, 1+\mathfrak{m}_{\mathbb{C}_K})$$

in the same way we define the map α. Since the natural map $T_p(G^\vee) \to T_p((G^\circ)^\vee)$ is surjective, we obtain a commutative diagram

$$\begin{array}{ccc} G^\circ(\mathcal{O}_{\mathbb{C}_K}) & \hookrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) \\ \downarrow \alpha^\circ & & \downarrow \alpha \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_p((G^\circ))) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) \end{array}$$

where both horizontal arrows are injective. In particular, we have $g \in \ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K)$. Moreover, we have $\ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K) = \ker(\alpha^\circ)^{\Gamma_K}$, which is a $\mathbb{Q}_p$-vector space since $\ker(\alpha^\circ)$ is a $\mathbb{Q}_p$-vector space by the same argument as in the preceding paragraph.
Therefore for every $n \in \mathbb{Z}$ there exists an element $g^n \in \ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K)$ with $g = p^n g_n$. However, this means $g = 0$ by lemma 1.3, yielding the desired contradiction.

Next we show that $d\alpha$ is injective on $t_G(K)$. Since $\log_G(G(\mathcal{O}_K)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = t_G(K)$ by Proposition from previous lecture, it is enough to show the injectivity on $\log_G(G(\mathcal{O}_K))$. Choose an arbitrary element $h \in G(\mathcal{O}_K)$ such that $\log_G(h) \in \ker(d\alpha)$. We wish to show that $\log_G(h) = 0$. As the isomorphism $\ker(\alpha) \cong \ker(d\alpha)$ in (3.8) is induced by $\log_G$, we can find $h' \in \ker(\alpha)$ with $\log_G(h) = \log_G(h')$.

Then by Proposition from last time we have $h - h' \in \ker(\log_G) = G(\mathcal{O}_{\mathbb{C}_K})_{\mathrm{tors}}$, which means that there exists some n with $p^n(h-h') = 0$, or equivalently $p^n h = p^n h'$. We thus find $p^n h \in \ker(\alpha) \cap G(\mathcal{O}_K)$, which implies $p^n h = 0$ by the injectivity of α on $G(\mathcal{O}_K)$.

Hence we have $h \in G(\mathcal{O}_{\mathbb{C}_K})_{\mathrm{tors}}$, thereby deducing $\log_G(h) = 0$. As $t_G(K) = t_G(\mathbb{C}_K)^{\Gamma_K}$ by Lemma 1.2, we can factor $d\alpha$ as

$$d\alpha : t_G(\mathbb{C}_K) \cong tG(K) \otimes_K \mathbb{C}_K \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K).$$

The first arrow is injective by our discussion in the preceding paragraph. The second arrow is injective since we have a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), K) \otimes_K \mathbb{C}_K$$

due to the freeness of $T_p(G^\vee)$ over $\mathbb{Z}_p$. Hence we deduce the injectivity of $d\alpha$ as desired, thereby completing the proof. □

Now we have the famous theorem from Tate:

Theorem 1.11 (Tate, 1967). *The maps α, $d\alpha$ from Proposition 1.10 induce isomorphisms on Γ_K-invariants:*

$$\alpha_K : G(\mathcal{O}_K) \to \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}),$$
$$d\alpha_K : t_G(\mathcal{O}_K) \to \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K).$$

Proof. By proposition 1.10, we have the following commutative diagram with the following exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \mathrm{coker}(\alpha) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \log_G} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & t_G(\mathbb{C}_K) & \xrightarrow{d\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & \mathrm{coker}(d\alpha) & \longrightarrow & 0
\end{array}$$

Applying $(\circ)^{\Gamma_K}$, we get a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & G(\mathcal{O}_K) & \xrightarrow{\alpha_K} & \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \mathrm{coker}(\alpha)^{\Gamma_K} \\
 & & \downarrow{\scriptstyle \log_G} & & \downarrow & & \downarrow{\scriptstyle \cong} \\
0 & \longrightarrow & t_G(K) & \xrightarrow{d\alpha} & \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & \mathrm{coker}(d\alpha)^{\Gamma_K}
\end{array}$$

By exactness, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{coker}(\alpha_K) & \hookrightarrow & \mathrm{coker}(\alpha)^{\Gamma_K} \\
\downarrow & & \downarrow \\
\mathrm{coker}(d\alpha_K) & \hookrightarrow & \mathrm{coker}(d\alpha)^{\Gamma_K}
\end{array}$$

Since $\mathrm{coker}(\alpha_K) \hookrightarrow \mathrm{coker}(d\alpha_K)$, it is enough to show that $d\alpha_K$ is surjective. Let

$$W = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K).$$

$$V = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K).$$

Then $d\alpha_K : t_G(K) \to V^{\Gamma_K}$. so $\dim_K(V_K^\Gamma) \geq \dim_K t_G(K) = \dim G = d$. We want to show that $\dim_K(V_K^\Gamma) = \dim_K(t_G(K))$. We also know that

$$\dim_K(W^{\Gamma_K}) \geq \dim_K(t_{G^\vee}(K)) = \dim(G^\vee) = d^\vee$$

and hence

$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \geq d + d^\vee = h.$$

It is therefore enough to show that

$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \leq h$$

Note that $\dim_{\mathbb{C}_K}(V) = h = \dim_{\mathbb{C}_K}(W)$. Recall that

$$T_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1))$$

as Γ_K=module, which induces a perfect Γ_K-equivalent pairing

$$T_p(G) \times T_p(G^\vee) \to \mathbb{Z}_p(1).$$

This gives a perfect Γ_K-equivariant pairing

$$V \times W \to \mathbb{C}_K(-1).$$

Taking Γ_K equivariant, we get

$$V^{\Gamma_K} \times W^{\Gamma_K} \to \mathbb{C}_K(-1)^{\Gamma_K} = 0.$$

This shows that $V^{\Gamma_K} \otimes_K \mathbb{C}_K$ and $W^{\Gamma_K} \otimes_K \mathbb{C}_K$ are orthogonal under this pairing. Hence

$$\dim_{\mathbb{C}_K}(V^{\Gamma_K} \otimes \mathbb{C}_K) + \dim_{\mathbb{C}_K}(W^{\Gamma_K} \otimes \mathbb{C}_K) \leq \dim_{\mathbb{C}_K}(V) = h,$$

completing the proof. □

Corollary 1.12. *We have that*

$$\dim(G) = \dim_K(\mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K)) = \dim_K(T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1))^{\Gamma_K}.$$

Proof. The first equality immediately follows from Theorem 3.4.10. The second equality follows by an identification

$$T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

where the isomorphisms are given by Proposition 1.7 and the freeness of $T_p(G^\vee)$ over $\mathbb{Z}_p$. □

We are finally ready to prove the first main result for this chapter.

Theorem 1.13 (Tate, 1967)**.** *Let G be a p-divisible group over $\mathcal{O}_K$. There is a canonical isomorphism of $\mathbb{C}_K[\Gamma_K]$-modules*

$$\mathrm{Hom}(T_p(G), \mathbb{C}_K) \cong t_{G^\vee}(\mathbb{C}_K) \oplus t^*_G(\mathbb{C}_K)(-1).$$

Proof. From the theorem 1.11, we have that

$$t_G(\mathbb{C}_K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K,$$

$$t_{G^\vee}(\mathbb{C}_K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K,$$

Moreover, from the proof of theorem 1.11 shows that $t_G(\mathbb{C}_K)$ and $t_{G^\vee}(\mathbb{C}_K)$ are orthogonal under the perfect pairing

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \times \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) \to \mathbb{C}_K(-1)$$

as constructed in the proof of theorem 1.11, with equality

$$\dim_{\mathbb{C}_K}(t_G(\mathbb{C}_K)) + \dim_{\mathbb{C}_K}(t_{G^\vee}(\mathbb{C}_K)) = \dim_{\mathbb{C}_K}(\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K)).$$

This means that $t_G(\mathbb{C}_K)$ and $t_{G^\vee}(\mathbb{C}_K)$ are orthogonal complements with respect to the above pairing, thereby yielding an exact sequence

$$0 \to t_{G^\vee}(\mathbb{C}_K) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \to t^*_G(\mathbb{C}_K)(-1) \to 0 \tag{1}$$

where for the last term we use the identification $\mathrm{Hom}_{\mathbb{C}_K}(t_G(\mathbb{C}_K), \mathbb{C}_K(-1)) \cong t^*_G(\mathbb{C}_K)(-1)$ that follows by observing that $t^*_G(\mathbb{C}_K)$ is the $\mathbb{C}_K$-dual $t_G(\mathbb{C}_K)$. Writing $d := \dim_{\mathbb{C}_K}(t_G(\mathbb{C}_K))$ and $d^\vee := \dim_{\mathbb{C}_K}(t_{G^\vee}(\mathbb{C}_K))$ we find

$$\mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(t^*_G(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) \cong \mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1)^{\oplus d^\vee}, \mathbb{C}_K^{\oplus d}) \cong H^1(\Gamma_K, \mathbb{C}_K(1))^{\oplus dd^\vee} = 0$$

thereby deducing that the exact sequence (1) above splits. Moreover, such a splitting is unique since we have

$$\mathrm{Hom}_{\mathbb{C}_K[\Gamma_K]}(t^*_G(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) \cong \mathrm{Hom}_{\mathbb{C}_k[\Gamma_K]}(\mathbb{C}_K(-1)^{\oplus d^\vee}, \mathbb{C}_K^{\oplus d}) \cong H^0(\Gamma_K, \mathbb{C}_K(1))^{\oplus dd^\vee} = 0$$

Hence we obtain the desired assertion. □

Definition 1.14. *Given a p-divisible group G over $\mathcal{O}_K$, we refer to the isomorphism in Theorem 1.13 as the Hodge-Tate decomposition for G.*

Corollary 1.15. *For every p-divisible group G over $\mathcal{O}_K$, the rational Tate-module*

$$V_p(G) := V_p(G \times_{\mathcal{O}_K} K) = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a Hodge-Tate p-adic representation of Γ_K.

Proof. As the $\mathbb{C}_K$-duals of $t_{G^\vee}(\mathbb{C}_K)$ and $t^*G(\mathbb{C}_K)$ are respectively given by $t^*_{G^\vee}(\mathbb{C}_K)$ and $t_G(\mathbb{C}_K)$, Theorem 1.13 yields a decomposition

$$V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong t^*_{G^\vee}(\mathbb{C}_K) \oplus t_G(\mathbb{C}_K)(1)$$

Then for each n we find

$$(V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n))^{\Gamma_K} \begin{cases} (t^*_{G^\vee}(\mathbb{C}_K) & \text{if } n = 0, \\ t_G(\mathbb{C}_K) & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}$$

by theorem from previous lecture. The assertion is now obvious by definition from the previous lectures. □

Proposition 1.16. *Suppose A is an abelian variety over K with good reduction. Then*

$$H^n_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(A, \Omega^j_{A/K}) \otimes_K \mathbb{C}_K(-j).$$

Proof. Since A has good reduction, there is an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$ such that the generic fiber is $\mathcal{A} \times K \cong A$. Moreover, we know that

$$\mathcal{A}^\vee[p^\infty] \cong \mathcal{A}[p^\infty]^\vee.$$

We have the following facts:

1. $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) = \text{Hom}_{\mathbb{Z}_p}(T_p(\mathcal{A}[p^\infty]), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$,
2. the formal completion of $\mathcal{A}$ at the unit element gives the formal group law corresponding to $\mathcal{A}[p^\infty]^\circ$ under the Serre-Tate equivalence
3. we have the isomorphism
$$H^0(A, \Omega^1_{A/K}) \cong t^*_e(A) \quad \text{and} \quad H^1(A, \mathcal{O}_A) \cong t_e(A^\vee)$$
where $t^*_e(A)$ and $t_e(A)$ respectively denote the cotangent space of A and tangent space of $A^\vee$ (at the unit section).
4. We have identifications
$$H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \bigwedge^n H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p),$$
$$H^i(A, \Omega^j_{A/K}) \cong \bigwedge^i H^1(A, \mathcal{O}_A) \otimes \bigwedge^j H^0(A, \Omega^1_{A/K}).$$
The statements (2) and (3) together yield identifications
$$H^0(A_{\overline{K}}, \Omega^j_{A/K}) \cong t^*_{\mathcal{A}[p^\infty]}(K) \quad \text{and} \quad H^1(A, \mathcal{O}_A) \cong t_{\mathcal{A}^\vee[p^\infty]}(K).$$
Hence Theorem 1.13 yields a canonical Γ_K-equivariant isomorphism
$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong (H^1(A, \mathcal{O}_A) \otimes_K \mathbb{C}_K) \oplus (H^0(A, \Omega^1_{A/K}) \otimes_K \mathbb{C}_K(-1)).$$
We then obtain the desired isomorphism by (4)

□

Proposition 1.16 is a special case of the general Hodge-Tate decomposition theorem that we introduced during the first week. The original proof by Faltings in [Fal88] relies on the language of almost mathematics. Recently, inspired by the work of Faltings, Scholze [Sch13] extended the Hodge-Tate decomposition theorem to rigid analytic varieties using his theory of perfectoid spaces. A good exposition of Scholze's work can be found in Bhatt's notes [Bha].

Corollary 1.17. *For every abelian variety A over K with good reduction, the étale cohomology $H^n_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_p)$ is a Hodge-Tate p-adic representation of* Γ_K.

For each j we have the identification

$$(H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(j))^{\Gamma_K} \cong \begin{cases} H^{n-j}(X, \Omega^j_{X/K}) & \text{if } 0 \leq j \leq n, \\ 0 & \text{otherwise} \end{cases}$$