p-adic Hodge Theory (Spring 2023): Week 6

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This week: Hodge-Tate Decomposition
In this section, we finally enter the realm of p-adic Hodge theory. Assuming some technical results from algebraic number theory, we prove two fundamental theorems regarding p-divisible groups, namely the Hodge-Tate decomposition for the Tate modules and the fullfaithfulness of the generic fiber functor. The primary reference for this section is Tate's paper [Tat67].

1 Formal point on p-divisible groups

For the rest of this section, we fix the base ring $R = \mathcal{O}_K$. We also let L be the p-adic completion of an algebraic extension of K, and denote by $\mathfrak{m}_L$ its maximal ideal. We are particularly interested in the case where $L = \mathbb{C}_K$.

Definition 1.1. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. We define the group of $\mathcal{O}_L$-valued formal points on G by*

$$G(\mathcal{O}_L) := \varprojlim_i G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) = \varprojlim \varinjlim G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

Example 1.2. $\mu_{p^\infty}(\mathcal{O}_L) \cong 1 + \mathfrak{m}_L$.

Remark 1.3. *The group of "ordinary" $\mathcal{O}_L$-valued points on μ_{p^∞} is given by*

$$\varinjlim_v \mu_{p^v}(\mathcal{O}_L) = \varinjlim_v \{x \in \mathcal{O}_L^\times : x^{p^v} = 1\}$$

which precisely consists of p-power torsion elements in $\mathcal{O}_L^\times$. We thus see that $\mu_{p^\infty}(\mathcal{O}_L)$ contains many "non-ordinary" points.

Proposition 1.4. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$.*

- *Writing $G_v = \mathrm{Spec}(A_v)$ for each v, we have an identification*

$$G(\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\varprojlim_v A_v, \mathcal{O}_L).$$

- *$G(\mathcal{O}_L)$ is a $\mathbb{Z}_p$-module with the torsion part given by*

$$G(\mathcal{O}_L)_{\mathrm{tors}} \cong \varprojlim_v \varinjlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

- *If G is étale, then $G(\mathcal{O}_L)$ is isomorphic to a torsion group $G(k_L)$ where k_L denotes the residue field of $\mathcal{O}_L$.*

Proof. Note that we have $\mathcal{O}_L = \varprojlim_i \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ by completeness of $\mathcal{O}_L$. We also have $\varprojlim A_v = \varprojlim_{i,v} A_v/\mathfrak{m}^i A_v$ since each A_v is $\mathfrak{m}$-adically complete for being finite free over $\mathcal{O}_K$ by a general fact as stated in [Sta, Tag 031B]. We thus obtain an identification

$$\begin{aligned} G(\mathcal{O}_L) &\cong \varprojlim_i \varinjlim_v \mathrm{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong \varprojlim_i \varinjlim_v \mathrm{Hom}_{\mathcal{O}_K}(A_v/\mathfrak{m}^i A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\ &\cong \varprojlim_i \mathrm{Hom}_{\mathcal{O}_K}(\varprojlim_v A_v/\mathfrak{m}^i A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\ &\cong \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\varprojlim_{i,v} A_v/\mathfrak{m}^i A_v, \varprojlim_i \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\ &\cong \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\varprojlim_v A_v, \mathcal{O}_L) \end{aligned}$$

as asserted in (1) □

Remark 1.5. *The formal scheme $G := Spf(\varprojlim A_v)$ carries the structure of a formal group induced by the finite flat $\mathcal{O}_K$-group schemes G_v. Moreover, we can write the identification in (1) as $G(\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_K\text{-}formal}(Spf(\mathcal{O}_L), G)$.*

Corollary 1.6. *Let G be a connected p-divisible group dimension d over $\mathcal{O}_K$. We have a canonical isomorphism of $\mathbb{Z}_p$-modules*

$$G(\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_K\text{-}cont}(\mathcal{O}_K[[t_1, \cdots, t_d]], \mathcal{O}_L)$$

where the multiplication by p on the target is induced by $[p]_{\mu(G)}$.

Above imply $G(\mathcal{O}_L) \cong \mathfrak{m}^d_{\mathcal{O}_L}$ as a set.

Proposition 1.7. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. Then we have an exact sequence*

$$0 \to G^\circ(\mathcal{O}_L) \to G(\mathcal{O}_L) \to G^{\acute{e}t}(\mathcal{O}_L) \to 0.$$

Proof. Let $G_v = \mathrm{Spec}(A_v)$, $G_v^\circ = \mathrm{Spec}(A_v^\circ)$, and $G_v^{\text{ét}} = \mathrm{Spec}(A_v^{\text{ét}})$. Let

$$\mathcal{A} = \varprojlim A_v, \quad \mathcal{A}^{\text{ét}} = \varprojlim A_v^{\text{ét}}.$$

This sequence is left exact since colimits and limits are both left exact. We need to show that $G(\mathcal{O}_L) \to G^{\text{ét}}(\mathcal{O}_L)$ is surjective, i.e. the map

$$\mathrm{Hom}_{\mathrm{cibt}}(\mathcal{A}, \mathcal{O}_L) \to \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}^{\text{ét}}, \mathcal{O}_L)$$

is surjective. Recall that

$$G^\circ(\mathcal{O}_L) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}_K[\![t_1, \cdots, t_d]\!], \mathcal{O}_L)$$

where $d = \dim(G)$. Moreover,

$$(\mathcal{A}^{\text{ét}} \otimes k)[\![t_1, \cdots, t_d]\!] \cong \mathcal{A} \otimes k$$

since over k the connected–étale sequence splits.
We get $f : \mathcal{A}^{\text{ét}}[\![t_1, \cdots, t_d]\!] \to A$ (by the same argument as in Serre–Tate). We claim that this map is an isomorphism.
For surjectivity, assume $\mathrm{coker}(f) \neq 0$. Then there exists a maximal ideal $\mathfrak{M}$ of A such that $\mathrm{coker}(f)_{\mathfrak{M}} \neq 0$. Hence $\mathrm{coker}(f) \otimes_{\mathcal{O}_K} k = 0$, so m coker(f) = coker(f), and hence

$$\mathrm{coker}(f)_{\mathfrak{M}} = \mathfrak{m}\,\mathrm{coker}(f)_{\mathfrak{M}} = \mathfrak{M}\,\mathrm{coker}(f)_{\mathfrak{M}}.$$

Since $\mathrm{coker}(f)_{\mathfrak{M}}$ is finitely-generated over $\mathcal{A}_{\mathfrak{M}}$, we are done by Nakayama's Lemma. For injectivity, let $I = (t_1, \cdots, t_d)$ and I be the image of I under f. We have a short exact sequence

$$0 \to \ker(f)/\ker(f) \cap I^j \to \mathcal{A}^{\text{ét}}[\![t_1, \cdots t_d]\!]/I^j \to \mathcal{A}/\tilde{I^j} \to 0,$$

so $\ker(f)/\ker(f) \cap I^j = 0$, showing that $\ker(f) \subseteq I^j$. Since $\bigcap I^j = 0$, this shows that $\ker(f) = 0$.
We have hence shown that f is an isomorphism. This gives a surjection $\mathcal{A} \to \mathcal{A}^{\text{ét}}$ which splits the embedding $\mathcal{A}^{\text{ét}} \to \mathcal{A}$. We hence get a splitting of

$$\mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}, \mathcal{O}_L) \to \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}^{\text{ét}}, \mathcal{O}_L),$$

showing this map is surjective. □

Corollary 1.8. *For all $x \in G(\mathcal{O}_L), p^n x \in G^\circ(\mathcal{O}_L)$ for some n.*

Proof. The group $G^{\text{ét}}$ is torsion. Hence for some some n, the image of $p^n x$ in $G^{\text{ét}}(\mathcal{O}_L)$ is trivial. We are hence done by the connected–étale sequence. □

Proposition 1.9. *If the field L is algebraically closed (e.g. $L = \mathbb{C}_K$), multiplication by p on $G(\mathcal{O}_L)$ is surjective.*

Proof. By the connected–étale sequence, can work on $G^\circ(\mathcal{O}_L)$ and $G^{\text{ét}}(\mathcal{O}_L)$ separately. Since $G^{\text{ét}}(\mathcal{O}_L) = G^{\text{ét}}(k_L)$, using equivalence to finite free $\mathbb{Z}_p$-modules, multiplication by p is surjective. The group $G^\circ(\mathcal{O}_L)$ is p-divisible by the p-divisibility of the corresponding p-divisible formal group μ. Surjectivity on $G^\circ(\mathcal{O}_L)$ follows. □

Remark 1.10. *These facts will imply that $\log : G(\mathcal{O}_{\mathbb{C}_K}) \to t_G(\mathbb{C}_K)$ is surjective.*

2 The logarithm for p-divisible groups

Definition 2.1. *Let G be a p-divisible group over $\mathcal{O}_K$ of dimension d. Let us write $\mathcal{A}^\circ := \mathcal{O}_K[\![t_1, \cdots, t_d]\!]$ and denote by $\mathcal{I}$ the augmentation ideal of $\mu(G)$.*

1. *Given an $\mathcal{O}_K$-module M, we define the tangent space of G with values in M by*

$$t_G(M) := \mathrm{Hom}_{\mathcal{O}_K\text{-}mod}(\mathcal{I}/\mathcal{I}^2, M).$$

and the contagent space of G with values in M by

$$t_G^*(M) := \mathcal{I}/\mathcal{I}^2 \otimes_{\mathcal{O}_K} M.$$

2. *We define the valuation filtration of $G^\circ(\mathcal{O}_L)$ by setting*

$$\mathrm{Fil}^\lambda\, G^\circ(\mathcal{O}_L); +\{f \in G^\circ(\mathcal{O}_L) : \nu(f(x)) \geq \lambda \text{ for all } x \in \mathcal{I}\}$$

for all real number $\lambda > 0$, where we identify $G^\circ(\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_K\text{-}cont}(\mathcal{A}^\circ, \mathcal{O}_L)$ as described in Corollary 1.6.

Remark 2.2. *We may identify t_G and t_G^* respectively with the tangent space and cotangent space of the formal group G_μ induced by μ.*

Lemma 2.3. *Let G be a p-divisible group over $\mathcal{O}_K$. For every $f \in \mathrm{Fil}^\lambda\, G^\circ(\mathcal{O}_L)$, we have $pf \in \mathrm{Fil}^\kappa\, G^\circ(\mathcal{O}_L)$ where $\kappa = \min(\lambda + 1, 2\lambda)$.*

Proof. Let $\mathcal{I}$ denote the augmentation ideal of $\mu(G)$. From lemma a few weeks ago we have that $[p]_{\mu(G)}(x) = px + y$ for some $y \in \mathcal{I}^2$. We thus find

$$(pf)(x) = f([p]_{\mu(G)}(x)) = f(px + y) = pf(x) + f(y),$$

which implies that $\nu((pf)(x)) \geq \min(\lambda + 1, 2\lambda)$. □

Lemma 2.4. *For every $x \in \mathcal{I}, f \in G(\mathcal{O}_L)$,*

$$\lim_{n\to\infty} \frac{(p^n f)(x)}{p^n}$$

exists in L and equal zero if $x \in \mathcal{I}^2$.

Proof. Recall that for any $f \in G(\mathcal{O}_L)$, $p^n f \in G^\circ(\mathcal{O}_L)$ for $n \gg 0$ by Corollary 1.8, hence we can apply lemma 2.3 to $p^n f \in G^\circ(\mathcal{O}_L)$.
By an easy induction, there exists c such that

$$p^n f \in \mathrm{Fil}^{n+c}\, G^\circ(\mathcal{O}_L) \text{ for } n \gg 0.$$

Indeed, if $\lambda \geq 1, \min(1+\lambda, 2\lambda) = 1 + \lambda$ and $\lambda < 1, \min(1+\lambda, 2\lambda) = 2\lambda$.
We now want to show that $(\frac{(p^n f)(x)}{p^n})$ is Cauchy. We have that

$$\begin{aligned}\frac{(p^{n+1} f)(x)}{p^{n+1}} - \frac{(p^n f)(x)}{p^n} &= \frac{(p^n f)([p]_\mu(x))}{p^{n+1}} - \frac{(p^n f)(px)}{p^{n+1}} \\ &= \frac{(p^n f)([p]_\mu(x) - px)}{p^{n+1}} \\ &= \frac{(p^n f)(y)}{p^{n+1}}\end{aligned}$$

has valuation $\geq 2(n+c) - (n+1) = n + 2c - 1$. This shows that the limit exists.
We finally want to show that the limit is 0 if $x \in \mathcal{I}^2$. By the same calculation as above,

$$v\left(\frac{(p^n f)(x)}{p^n}\right) \geq 2(n+c) - n \geq n + 2c,$$

so the sequence tends to 0. □

So from this, we would able to make the following definition

Definition 2.5. *Let G be a p-divisible group over $\mathcal{O}_K$, and let $\mathcal{I}$ denote the augmentation ideal of $\mu(G)$. We define the logarithm of G to be the map*

$$\log_G : G(\mathcal{O}_L) \to t_G(L)$$

such that for every $f \in G(\mathcal{O}_L)$, and that $x \in \mathcal{I}/\mathcal{I}^2$, we have

$$\log_G(f)(x) = \varinjlim_{n\to\infty} \frac{(p^n)(\tilde{x})}{p^n}$$

where $\tilde{x}$ is any lift of x to $\mathcal{I}$.

Example 2.6. *Let us provide an explicit description of $\log_{\mu_{p^\infty}}$. As seen in the previous weeks, we have that $\mu_{\widehat{\mathbb{G}}_m}[p^\infty] \cong \mu_{p^\infty}$. Corollary 1.6 gives an identification*

$$\mu_{p^\infty}(O_L) \cong \mathrm{Hom}_{\mathcal{O}_K\text{-}cont}(\mathcal{O}_L[\![t]\!], \mathcal{O}_L) \cong \mathfrak{m}_L \cong 1 + \mathfrak{m}_L.$$

We thus have the following commutative diagram:

$$\begin{array}{ccc} \mu_{p^\infty}(\mathcal{O}_L) & \xrightarrow{\log_{\mu_{p^\infty}}} & t_{\mu_{p^\infty}}(L) \\ {\scriptstyle f\mapsto 1+f(t)}\downarrow\wr & & \wr\downarrow{\scriptstyle g\mapsto g(t)} \\ 1+\mathfrak{m}_L & \longrightarrow & L \end{array} \tag{3.4}$$

Let us identify $\log_{\mu_{p^\infty}}$ with the bottom arrow. We also take an arbitrary element $1 + x \in 1 + \mathfrak{m}_L$. As each $f \in \mu_{p^\infty}(\mathcal{O}_L)$ satisfies

$$(p^n f)(t) = f\left([p^n]_{\widehat{\mathbb{G}}_m}(t)\right) = f((1+t)^{p^n} - 1) = (1 + f(t))^{p^n} - 1,$$

the diagram 3.4 yields an expression

$$\log_{\mu_{p^\infty}}(1+x) = \lim_{n\to\infty} \frac{(1+x)^{p^n} - 1}{p^n} = \lim_{n\to\infty} \sum_{i=1}^{p^n} \frac{1}{p^n}\binom{p^n}{i} x^i. \tag{1}$$

In addition, for eahc i and n we have that

$$\frac{1}{p^n}\binom{p^n}{i} - \frac{(-1)^{i-1}}{i} = \frac{(p^n-1)\cdots(p^n-i+1) - (-1)^{i-1}(i-1)!}{i!}.$$

Since the numerator is divisible by p^n, we obtain an estimate

$$\nu\left(\frac{1}{p^n}\binom{p^n}{i}x^i - \frac{(-1)^{i-1}x^i}{i}\right) \geq n + i\nu(x) - \nu(i!) \geq n + i\nu(x) - \frac{i}{p-1}.$$

Hence we may write the expression (1) as

$$\log_{\mu_{p^\infty}}(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i.$$

which coincides with the p-adic logarithm.

Let us collect some basic properties of the logarithm for p-divisible groups.

Proposition 2.7. *Let G be a p-divisible group over $\mathcal{O}_K$. Denote by $\mathcal{I}$ the augmentation ideal of $\mu(G)$.*

1. *$\log_G$ is a group homomorphism*
2. *$\log_G$ is a local isomorphism in the sense that for each real number $\lambda \geq 1$, it induces an isomorphism*
$$\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \xrightarrow{\sim} \{\tau \in t_G(L) : \nu(\tau(x)) \geq \lambda \text{ for all } x \in \mathcal{I}/\mathcal{I}^2\}.$$
3. *The kernel of $\log_G$ is the torsion subgroup $G(\mathcal{O}_L)_{tors}$ of $G(\mathcal{O}_L)$.*
4. *$\log_G$ induces an isomorphism $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong t_G(L)$.*

Proof. We first check (1). For all $f, g \in G(\mathcal{O}_L)$, we want to show that

$$\log_G(f+g) = \log_G(f) + \log_G(g).$$

We have that

$$\frac{p^n(f+g)(x)}{p^n} = \frac{(p^n f \otimes p^n g)(\mu(x))}{p^n} = \frac{(p^n f)(x) + (p^n g)(x) + y}{p^n}$$

for all $y \in (p^n f)I \otimes (p^n g)(I)$.

Since the valuation of y gets really large as $n \to \infty$, this shows that

$$\frac{p^n(f+g)(x)}{p^n} - \frac{(p^n f)(x)}{p^n} - \frac{(p^n g)(x)}{p^n} \to 0.$$

Let us now fix an arbitrary real number $\lambda \geq 1$ and write

$$\operatorname{Fil}^\lambda t_G(L) := \{\tau \in t_G(L) : \nu(\tau(x)) \geq \lambda \text{ for all } x \in \mathcal{I}/\mathcal{I}^2\}.$$

If $f \in \operatorname{Fil}^\lambda G^\circ(\mathcal{O}_L)$, then lemma 2.3 implies that $\nu(\frac{(p^n f)(x)}{p^n}) \geq \lambda$ for all $x \in \mathcal{I}$ and that $n > 0$, thereby implying $\log_G(f) \in \operatorname{Fil}^\lambda t_G(L)$. It is then straightforward to verify that $\log_G$ on $\operatorname{Fil}^\lambda G^\circ(\mathcal{O}_L)$ admists an inverse $\operatorname{Fil}^\lambda t_G(L) \to \operatorname{Fil}^\lambda G^\circ(\mathcal{O}_L)$ which sends each $\tau \in \operatorname{Fil}^\lambda t_G(L)$ to the unique $f \in \operatorname{Fil}^\lambda G^\circ(\mathcal{O}_L)$ with $f(t_i) = \tau(t_i)$. Therefore we deduce (2).

Next we show $\ker(\log_G) = G(\mathcal{O}_L)_{tors}$ as asserted in (3). We clearly have $G(\mathcal{O}_L)_{tors} \subseteq \ker(\log_G)$ since $t_G(L)$ is torsion free for being a vector space over L. Hence we only need to establish the reverse inclusion $\ker(\log_G) \subset G(\mathcal{O}_L)$. Let f be an element in $\ker(\log_G)$. By (1) we have $p^n f \in \ker(\log_G)$ for all n. Moreover, Corollary 1.8 and Lemma 2.3 together yield $p^n f \in \operatorname{Fil}^1 G^\circ(\mathcal{O}_L)$ for all sufficiently large n. We then find $p^n f = 0$ for all sufficiently large n by (2), thereby deducing that f is a torsion element as desired.

Now (3) readily implies the injectivity of the map $G(\mathcal{O}_L) \otimes \mathbb{Z}_p\mathbb{Q}_p \to t_G(L)$ induced by $\log_G$. We also deduce the surjectivity of the map from (2) by observing that every element $\tau \in t_G(L)$ satisfies $p^n\tau \in \operatorname{Fil}^1 t_G(L)$ for all sufficiently large n.

□