p-adic Hodge Theory (Spring 2023): Week 5

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This week: Serre-Tate Equivalence & Dieudonne-Manin Classification

1 Dieudonné–Manin classification

Let k be a perfect field of characteristic p. Let σ be the Frobenius automorphism over k.

Definition 1.1. *We write $W(k)$ for the ring of Witt vectors over k. We write $K_0(k)$ for the fraction field of $W(k)$. The Frobenius $\sigma_{W(k)}$ on $W(k)$ is*

$$\sigma\left(\sum_{n\geq 0}\tau(x_n)p^n\right)=\sum_{n\geq 0}\tau(x_n^p)p^n$$

where $\tau : k \to W(k)$ is the Teichmuller lift. Finally, $\sigma_{K_0}(k)$ is the unique field of automorphism on $K_0(k)$ extending $\sigma_{W(k)}$.

Example 1.2. *Let $k = \mathbb{F}_q$ and ζ_{q-1} be a primitive $(q-1)$st root of unity. Then*

$$W(k) = \mathbb{Z}_p[\zeta_{q-1},\ K_0(k) = \mathbb{Q}_p[\zeta_{q-1}]$$

and σ acts on $W(k)$ by

$$\sigma(\zeta_{q-1}) = \zeta_{q-1}^p,$$

and trivially on $\mathbb{Z}_p$.

Definition 1.3. *A Dieudonné module over k is a pair (M, φ) where*

1. *M is a finite free module over $W(k)$,*
2. *$\varphi : M \to M$ is an additive map such that:*
   (a) *φ is σ-linear, i.e. $\varphi(am) = \sigma(a)\varphi(m)$ for all $a \in W(k), m \in M$,*
   (b) *$\varphi(M) \supseteq pM$.*

Theorem 1.4 (Dieudonné)**.** *There is an anti-equivalence:*

$$\mathbb{D} :\{p\text{-divisible groups over } k\} \to \{\text{Dieudonné modules over } k\}$$

such that

1. *$rk(\mathbb{D}(G)) = ht(G)$,*
2. *G is étale if and only if $\varphi_{\mathbb{D}(G)}$ is an isomorphism.*
3. *G is connected if and only if $\varphi_{\mathbb{D}(G)}$ is topologically nilpotent,*
4. *$[p]_G$ induces multiplication by p on $\mathbb{D}(G)$.*

We shall omit a proof here. But the interested reader could read [Dem86].

Remark 1.5. *There is a notion of duality for Dieudonné modules, compatible with Cartier duality.*

Example 1.6. *Let σ denote the Frobenius automorphism of $W(k)$.*

- *$\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})$ is isomorphic to $W(k)$ together with $\varphi_{\mathbb{D}}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}) = \sigma$.*

- $\mathbb{D}(\mu_{p^\infty})$ *is* $W(k)$ *together with* $\varphi_{\mathbb{D}}(\mu_{p^\infty}) = p\sigma$.
- *If* E *is an ordinary elliptic curve over* $\overline{k}$, *we have* $\mathbb{D}(E[p^\infty]) \cong w(\overline{k})^{\oplus 2}$ *together with* $\varphi_{\mathbb{D}}(E[p^\infty]) = \sigma \oplus p\sigma$.

Definition 1.7. *A map of p-divisible groups* $f : G \to H$ *is an isogeny if it is surjective and* $\ker f$ *is finite flat.*

Proposition 1.8. *A homomorphism* $f : G \to H$ *of p-divisible groups over* k *is an isogeny if and only if the following equivalent conditions are satisfied:*

1. *The induced map* $\mathbb{D}(H) \to \mathbb{D}(G)$ *is injective.*
2. *THe induced map* $\mathbb{D}(H)[1/p] \to \mathbb{D}(G)[1/p]$ *is an isomorphism.*

Definition 1.9. *An isocrystal over* k *is a finite-dimensional* $K_0(k)$*-vector space* N *with a* σ*-linear bijection* $\varphi : N \to N$.

Remark 1.10. *If* G *is a p-divisible group over* k, $D(G)[1/p]$ *is an isocrystal which determines the isogeny class of* G.

Example 1.11. *Let* $\lambda \in \mathbb{Q}$ *be* $\lambda = \frac{d}{r}$ *with* $(d, r) = 1$, $r > 0$. *The simple isocrystal* $N(\lambda)$ *of slope* λ *is the* $K_0(k)^{\oplus r}$ *space with the* σ*-semilinear automorphism* $\varphi_{N(\lambda)}$ *given by*

$$\varphi_{N(\lambda)}(e_1) = e_2, \quad \varphi_{N(\lambda)}(e_2) = e_3, \quad \cdots \varphi_{N(\lambda)}(e_{r-1}) = e_r, \varphi_{N(\lambda)}(e_r) = p^d e_1,$$

where $e_1, \cdots, e_r$ *denote the standard basis vectors. It is straightforward to verify that* $N(\lambda)$ *is of rank* r, *degree* d.

Theorem 1.12. *Let* $k = \overline{k}$. *The category of isocrystals over* k *is semisimple with simple objects given by* $N(\lambda)$. *In other words, any* N *over* k *has a decomposition*

$$N = \bigoplus_{i=1}^{\ell} N(\lambda_i)^{\oplus m_i},$$

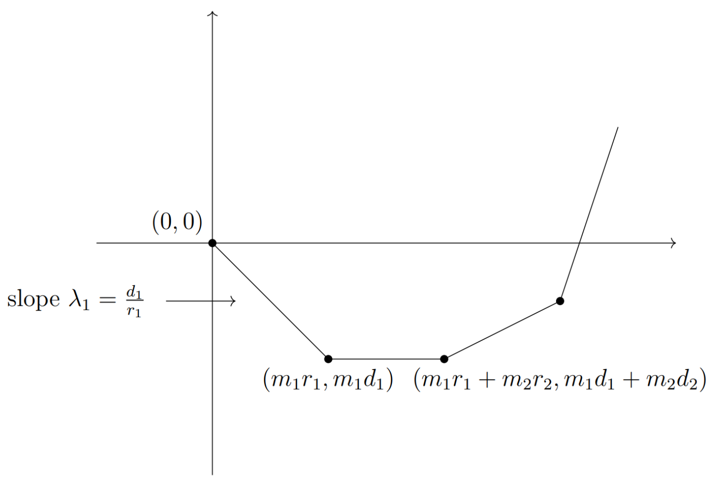
for $\lambda_1 < \cdots < \lambda_\ell$.

Definition 1.13. *For* $\lambda_i = \frac{d_i}{r_i}, (d_i, r_i) = 1, r_i > 0$.

1. *The Newton polygon of* N *is the lower convex hull of the points*

$$(m_1 r_1 + \cdots m_i r_i, m_1 d_1 + \cdots + m_i d_i), i = 1, 2, \cdots, \ell$$

Here is a schematic diagram of N:



2. *The dimension of* N *is* $\dim(N) = m_1 d_1 + m_2 d_2 + \cdots + m_\ell d_\ell$.
3. *The slope of* N *is* $\mu(N) = \frac{\dim(N)}{\text{rank}(N)}$.

Proposition 1.14. *If G is a p-divisible group over $\overline{k}$, then $\mathbb{D}(G)[1/p]$ has rank $ht(G)$ and dimension $\dim(G)$. Moreover, if*

$$\mathbb{D}(G)][1/p] = \bigoplus_{i=1}^{\ell} N(\lambda_i)^{\oplus} m_i,$$

then

$$\mathbb{D}(G^{\vee})[1/p] = \oplus N(1-\lambda_i)^{\oplus m_i}.$$

WE note the following proposition:

Proposition 1.15. *Let N be an isocrystal over K0(k). Then we have $N \cong \mathbb{D}(G)[1/p]$ for some p-divisible group G of height h and dimension d over k if and only if the following conditions are satisfied:*

1. *N is of rank h and degree d.*
2. *Every Newton slope λ of N satisfies $0 \leq \lambda \leq 1$.*

Theorem 1.16 (Serre, Honda-Tate, Oort). *Let N be an isocrystal over $K_0(\overline{k})$. Then*

$$N \cong \mathbb{D}(A[p^{\infty}])[1/p]$$

for some abelian variety A over $\overline{k}$ of dimension g if and only if the following conditions are satisfied:

1. *N is of rank $2g$ and degree g.*
2. *Every Newton slope λ of N satisfies $0 \leq \lambda \leq 1$.*
3. *If $\lambda \in \mathbb{Q}$ occurs as a Newton slope of N, then $1-\lambda$ occurs as a Newton slope of N with the same multiplicity.*

Remark 1.17. *The necessity part is easy to verify by Proposition 1.15. The main difficulty lies in proving the sufficiency part, which was initially conjectured by Manin [Man63].*

Definition 1.18. *Let A be a principally polarized abelian variety of dimension g over $\overline{k}$.*

1. *We define its Newton polygon by Newt(A) := Newt($\mathbb{D}(A[p^{\infty}])[1/p]$).*
2. *We say that A is ordinary if Newt(A) connects the points $(0,0), (g,0)$, and $(2g,g)$.*
3. *We say that A is supersingular if Newt(A) connects the points $(0,0)$ and $(2g,g)$.*

Example 1.19. *Let A be an ordinary abelian variety of dimension g over k. A priori, this means that there exists an isogeny $A[p^{\infty}] \to \mu_{p^{\infty}}^g \times (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g$. We assert that there exists an isomorphism*

$$A[p\infty]'\mu gp\infty \times (\mathbb{Q}_p/\mathbb{Z}_p)g.$$

By Proposition lst time, an exact sequence

$$0 \to A[p^{\infty}]^{\circ} \to A[p^{\infty}] \to A[p^{\infty}]^{\acute{e}t} \to 0.$$

Moreover, this sequence splits as it splits at every finite level. Hence we have a decomposition

$$A[p^{\infty}] \cong A[p^{\infty}]^{\circ} \times A[p^{\infty}]^{\acute{e}t}.$$

Proposition 1.15 implies that $A[p^{\infty}]^{\acute{e}t}$ should correspond to the slope 0 part of Newt(A), and thus have height g. We then deduce $A[p^{\infty}]^{\acute{e}t} \cong (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g$ by the remark after Proposition 1.18, and $A[p^{\infty}]^{\circ} \cong (A[p^{\infty}]^{\acute{e}t})^{\vee} \cong \mu_{p^{\infty}}^g$ by self-duality of $A[p^{\infty}]$.

Proposition 1.20. *For an abelian variety A over k, there exist a natural identification*

$$H^1_{\mathrm{cris}}(A/W(k)) \cong \mathbb{D}(A[p^{\infty}]).$$

Remark 1.21. *In light of the crystalline comparison theorem as introduced in week 1, this identification provides a powerful tool to study abelian varieties and their moduli spaces, such as (local) Shimura varieties of PEL (polarization, endomorphism, and level structure) or Hodge type, using p-adic Hodge theory and the theory of Dieudonné modules/isocrystals.*

2 Hodge-Tate Decomposition

In this section, we finally enter the realm of p-adic Hodge theory. Assuming some technical results from algebraic number theory, we prove two fundamental theorems regarding p-divisible groups, namely the Hodge-Tate decomposition for the Tate modules and the fullfaithfulness of the generic fiber functor. The primary reference for this section is Tate's paper [Tat67].

2.1 The completed algebraic closure of p-adic field

Definition 2.1. *Let K be extension of $\mathbb{Q}_p$ with a nonarchimedean valuation ν.*

1. *We define the valuation ring of K by $\mathcal{O}_K := \{x \in K : \nu(x) \geq 0\}$.*
2. *K is a p-adic field if it is discrete valued and complete with a perfect residue field.*

Example 2.2.

1. *Every finite extension of $\mathbb{Q}_p$ is a p-adic field.*
2. *Given a perfect field k of characteristic p, the fraction field of the ring of Witt vectors $W(k)$ is a p-adic field.*

Remark 2.3. *The fraction field of $W(\mathbb{F}_p)$ is the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. Hence it is a p-adic field which is not an algebraic extension of $\mathbb{Q}_p$.*

For the rest of this week, let K be p-adic field with abosolute Galois group Γ_K. Let $\mathfrak{m}$, k be maximum ideal/residue field of $\mathcal{O}_K$.

Definition 2.4. *The completed algebraic closure of K is defined to be $\mathbb{C}_K : \widehat{\overline{K}}$.*

So clearly $\mathbb{C}_K$ is not p-adic field, as its valuation is not discrete: It is first example of characteristic 0 perfectoid field.

Lemma 2.5. *The action of Γ_K on K uniquely extends to a continuous action on $\mathbb{C}_K$.*

Now let's fix a valuation ν on $\mathbb{C}_K$ with $\nu(p) = 1$.

Proposition 2.6. *The field $\mathbb{C}_K$ is algebraically closed.*

Proof. Let $p(t)$ be an arbitrary non-constant polynomial over $\mathbb{C}_K$. May assume WLOG $p(t)$ is monic over $\mathcal{O}_{\mathbb{C}_K}$ by scaling, so we can write

$$p(t) = t^d + a_1 t^{d-1} + \cdots + a_d$$

for some $a_i \in \mathcal{O}_{\mathbb{C}_K}$. For each n, choose a polynomial

$$p_n(t) = t^d + a_{1,n} t^{d-1} + \cdots + a_{d,n}$$

with $a_{i,n} \in \mathcal{O}_{\mathbb{C}_K}$. For each n, we choose a polynomial

$$p_n(t) = t^d + a_{1,n} t^{d-1} + \cdots + a_{d,n},$$

with $a_i \in \mathcal{O}_{\overline{K}}$ and that $\nu(a_i - a_{i,n}) \geq dn$.
Let us choose $\alpha_1 \in \mathcal{O}_{\overline{K}}$ with $p_1(\alpha_1) = 0$. We proceed by induction on n to choose $\alpha_n \in \mathcal{O}_{\overline{K}}$ with $p_n(\alpha_n) = 0$ and $\nu(\alpha_n - \alpha_{n-1}) \geq n - 1$. Since $a_{i,n} - a_{i,n-1} = (a_{i,n} - a_i) + (a_i - a_{i,n-1})$ has valuation at least $d(n-1)$, we find $\nu(p_n(\alpha_{n-1})) \geq d(n-1)$ by observing

$$p_n(\alpha_{n-1}) = p_n(\alpha_{n-1}) - p_{n-1}(\alpha_{n-1}) = \sum_{i=1}^{d} (a_{i,n} - a_{i,n-1}) \alpha_{n-1}^{d-i}.$$

Moreover, we have

$$p_n(\alpha_{n-1}) = \prod_{i=1}^{d} (\alpha_{n-1} - \beta_{n,i})$$

where $\beta_{n,1}, \cdots, \beta_{n,d}$ are roots of $p_n(t)$. Note that $\beta_{n,i} \in \mathcal{O}_{\overline{K}}$ since $\mathcal{O}_{\overline{K}}$ is integrally closed. As $\nu(p_n(\alpha_{n-1})) \geq d(n-1)$, we deduce that $\nu(\alpha_{n-1}, \beta_{n,i}) \geq n - 1$ for some i. We thus complete the induction step by taking $\alpha_n := \beta_{n,i}$.
Since the sequence (α_n) is Cauchy by construction, it converges to an element $\alpha \in \mathcal{O}_{\mathbb{C}_K}$. Moreover, for each n we find $\nu(p(\alpha_n)) \geq dn$ by observing

$$p(\alpha_n) = p(\alpha_n) - p_n(\alpha_n) = \sum_{i=1}^{d} (a_i - a_{i,n}) \alpha_n^{d-i}.$$

We thus have $p(\alpha) = 0$, thereby completing the proof. □

Definition 2.7. *A p-adic representation of Γ_K is a finite dimensional $\mathbb{Q}_p$-vector space V together with a continuous homomorphism $\Gamma_K \to GL(V)$. We denote by $Rep_{\mathbb{Q}_p}(\Gamma_K)$ the category of p-adic Γ_K-representations.*

Example 2.8. *every p-divisible group G over K gives rise to a p-adic Γ_K representation $V_p(G) := T_p(G) \otimes \mathbb{Z}_p \mathbb{Q}_p$, called the rational Tate module of G*

Definition 2.9 (Tate Twist)**.** *Given a $\mathbb{Z}_p[\Gamma_K]$-module M, we define its n-th Tate twist to be the $\mathbb{Z}_p[\Gamma_K]$-module*

$$M(n) : \begin{cases} M \otimes_{\mathbb{Z}_p} T_p(\mu_{p^\infty})^{\otimes n} & \text{if } n \geq 0, \\ \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(\mu_{p^\infty})^{\otimes -n}, M) & \text{if } n < 0, \end{cases}$$

Example 2.10. *By definition, we have $\mathbb{Z}_p(1) = T_p(\mu_{p^\infty}) = \varprojlim \mu_{p^v}(\overline{K})$. The homomorphism $\chi_K : \Gamma_K \to Aut(\mathbb{Z}_p(1)) \cong \mathbb{Z}_p^\times$ which represents the Γ_K-action on $\mathbb{Z}_p(1)$ is called the p-adic cyclotomic character of K. We will often simply write χ instead of χ_K to ease the notation.*

Lemma 2.11. *Let M be a $\mathbb{Z}_p[\Gamma_K]$-module. For each $m, n \in \mathbb{Z}$, we have canonical Γ_K equivariant isomorphisms*

$$M(m) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n) \cong M(m+n) \quad \text{and} \quad M(n)^\vee \cong M^\vee(-n).$$

Lemma 2.12. *Let M be a $\mathbb{Z}_p[\Gamma_K]$-module, and let $\rho : \Gamma_K \to Aut(M)$ be the homomorphism which represents the action of Γ_K on M. For every $n \in \mathbb{Z}$ the action of Γ_K on $M(n)$ is represented by $\chi_n \cdot \rho$.*

Proof. Upon choosing a basis element e of $\mathbb{Z}_p(n)$, we obtain an isomorphism $M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n) \xrightarrow{\sim} M$ given by $m \otimes e \mapsto m$. The assertion now follows by observing that the Γ_K-action on $M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$ is given by $\rho \otimes \chi^n$. □

We assume the following fundamental result about the Galois cohomology of the Tate twists of $\mathbb{C}_K$

Theorem 2.13 (Tate-Sen)**.** *We have canonical isomorphisms*

$$H^i(\Gamma_K, \mathbb{C}_K(n)) \cong \begin{cases} K & \text{if } i = 0 \text{ or } 1, n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We now introduce the first class of p-adic Γ_K-representations.

Lemma 2.14 (Serre-Tate)**.** *For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, the natural $\mathbb{C}_K$-linear map*

$$\tilde{\alpha_V} : \bigoplus_{n\in\mathbb{Z}} \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K \mathbb{C}_K(n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

is Γ_K-equivalent and injective.

Proof. For each $n \in \mathbb{Z}$, we have a Γ_K-equivalent K-linear map

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n))^{\Gamma_K} \otimes_K K(n) \hookrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n) \otimes_K K(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

which give rise to a Γ_K-equivalent $\mathbb{C}_K$ linear map

$$\tilde{\alpha}_V^{(n)} : (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n))^{\Gamma_K} \otimes_K \mathbb{C}_K(n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K \qquad (*)$$

by extension of scalars. Hence we deduce that $\tilde{\alpha}_V = \bigoplus_{n\in\mathbb{Z}} \tilde{\alpha}_V^{(n)}$ is Γ_K-equivalent.
Now for each $n \in \mathbb{Z}$, choose basis $(v_{m,n})$ of $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n))^{\Gamma_K} \otimes_K K(n)$ over K. We may regard $v_{m,n}$ as a vector in $V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ via the map (*). Moreover, the source of the map $\tilde{\alpha}_V$ is spanned by the vectors $(v_{m,n})$.

If $\ker(\tilde{\alpha}_V)$ non-trivial, then exists nontrivial relation $\sum c_{m,n} v_{m,n} = 0$. Chose one with min length, WLOG suppose $c_{m_0,n_0} = 1$ for some m_0 and n_0. For every $\gamma \in \Gamma_K$ we find

$$0 = \gamma(\sum c_{m,n} v_{m,n}) - \chi(\gamma)^{n_0}(\sum c_{m,n} v_{m,n}) = \sum (\gamma(c_{m,n})\chi(\gamma)^n - \chi(\gamma)^{n_0} c_{m,n}) v_{m,n}$$

by Γ_K-equivariance of $\tilde{\alpha}_V$ and Lemma 2.11. Note that the coefficient of v_{m_0,n_0} in the last expression is 0. Hence the minimality of our relation implies that all coefficients in the last expression must vanish, thereby yielding relations

$$\gamma(c_{m,n})\chi(\gamma)^{n-n_0} = c_{m,n} \quad \text{for all } \gamma \in \Gamma_K.$$

Then by Lemma 2.11 and Theorem 2.12 we find $c_{m,n} = 0$ for $n \neq n_0$ and $c_{m,n} \in K$ for $n = n_0$. Therefore our relation $\sum c_{m,n} v_{m,n} = 0$ becomes a nontrivial K-linear relation among the vectors v_{m,n_0}, thereby yielding a desired contradiction. □

Definition 2.15. *We say that* $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ *is Hodge-Tate if the map* $\tilde{\alpha}_V$ *in lemma 2.13 is an isomorphism.*

We will see next (or next next) week that p-adic representations discussed in example 2.7 are Hodge-Tate in many cases.