p-adic Hodge Theory (Spring 2023): Week 4

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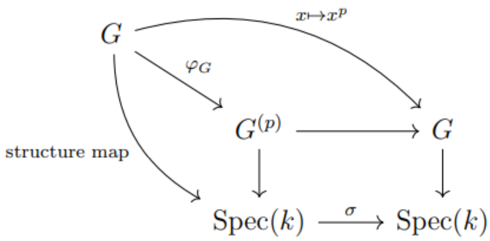
This week: p-divisible groups

1 From last time: Frobenius morphism

Let $R = k$ be a perfect field of characteristic p. Let σ be the Frobenius on k.

Definition 1.1. *Let $G = \mathrm{Spec}(A)$ be a finite k-group. The Frobenius twist is $G(p) = G \times_{k,\sigma} k$ and the (relative) Frobenius φG of G (over k) is defined by the diagram:*



More generally,

$$G^{(p^r)} = (G^{(p^{r-1})})^{(p)}$$

$$\varphi_G^r = \varphi_{G^{(p^{r-1})}} \circ \varphi_G^{r-1}$$

The Verschiebung of G is $\psi_G = \varphi_{G^\vee}^\vee$, where

$$\varphi_{G^\vee} : G^\vee \to (G^\vee)^{(p)}.$$

Remark 1.2. *Verschiebung ψ_G is a map $G^{(p)} \cong ((G^\vee)^{(p)})^\vee \to G$*

Remark 1.3. *We can check if a finite flat R-group scheme is connected or étale by passing to the special fiber. There are criteria for connected or étaleness for G_K in terms of Frobenius and the Verschiebung.*

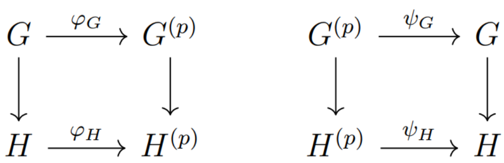
Lemma 1.4.

- *The Frobenius φ_G induces a map*

$$A^{(p)} = A \otimes_{k,\sigma} k \to A$$

$$a \otimes c \mapsto c \cdot a^p.$$

- *For any morphism $G \to H$ as schemes, we have induced maps*



- *Both φ_G and ψ_G are group homomorphisms.*

Recall that: for ring R of characteristic p, $\alpha_p := \mathrm{Spec}(R[t]/t^p)$.

Also, the nth roots of unit over R is $\mu_n = \mathrm{Spec}(R[t]/(t^n - 1))$. For any R-algebra B, $\mu_n(B) = \{b \in B | b^n = 1\}$.

And also: $\underline{M} := \amalg_{m\in M} \mathrm{Spec}(R) \cong \mathrm{Spec}(\prod_{m\in M} R)$.

Example 1.5. *We have that:*

1. $\varphi_{\alpha_p} = 0$, $\psi_{\alpha_p} = 0$.
2. $\varphi_{\underline{\mathbb{Z}/p\mathbb{Z}}}$ *is an isomorphism,* $\psi_{\underline{\mathbb{Z}/p\mathbb{Z}}} = 0$.
3. $\varphi_{\mu_p} = 0$, ψ_{μ_p} *is an isomorphism.*

Proposition 1.6. *We have that*

$$\psi_G \circ \varphi_G = [p]_G, \quad \varphi_G \circ \psi_G = [p]_{G^{(p)}}.$$

We shall omit the proof here, but interested reader may check Serin Hong's notes and Richard Pink's notes.

Proposition 1.7. *Suppose G is a finite group scheme over k. Then G is connected if and only if $\varphi_G^r = 0$ for some r. Moreover, G is étale if and only if φ_G is an isomorphism.*

Proof. If G is connected, A is a local Artinian ring. It decomposes as $A = k \oplus I$ where $I = \ker(\varepsilon)$. Since I is a maximal ideal, it is nilpotent, so there is $r > 0$ such that for all $x \in I, x^{p^r} = 0$. This shows that φ_G^r factors through the unit section.

Conversely, suppose $\varphi_G^r = 0$ for some r. Since φ_G^r induces an isomorphism $G(\overline{k}) \cong G^{(p^r)}(\overline{k})$, we have that $G(\overline{k}) = 0$, so G is connected.

If G is étale, $\ker(\varphi_G)$ is connected, so $\ker(\varphi_G) \subseteq G^0 = 0$. This shows that φ_G is injective. In fact, it is an injective homomorphism $\varphi_G : G \to G^{(p)}$ between groups of the same order, so it is an isomorphism.

Suppose now that φ_G is an isomorphism. It induces an isomorphism on G^0. Hence φ_{G^0} is an isomorphism, and hence $\varphi_{G^0}^r$ is an isomorphism. Since $\varphi_{G^0}^r = 0$ at some point (G^0 is connected), we see that $G^0 = 0$, and hence G is étale. □

Proposition 1.8. *Suppose G is a connected finite flat k-group. Then the order of G is a power of p.*

Proof. Let n be the order of G. We induct on n. As usual, let $I = \ker(\varepsilon)$ be the augmentation ideal. Choose $x_1, \cdots, x_d \in I$ which lifts a basis of I/I^2. Since G is connected, $d > 0$.

Then A be a local ring with maximal ideal I.

Let $H = \ker(\varphi_G)$. We first claim that the order of H is p^d. By Nakayama, $x_1, \cdots, x_d$ generate I. Hence

$$H = \mathrm{Spec}(A/(x_1^p, \cdots, x_d^p)).$$

We want to show that

$$\lambda : k[t_1, \cdots t_d]/(t_1^p, \cdots, t_d^p) \xrightarrow{\cong} A/(x_1^p, \cdots, x_d^p).$$

Surjectivity is clear. We have a natural map

$$\pi : A = k \oplus I \to I/I^2.$$

For each $j = 1, \cdots, d$, define $D_j : A \to A$ as the composition

$$A \xrightarrow{\mu} A \otimes A \xrightarrow{(id,\pi)} A \otimes_k I/I^2 \xrightarrow{x_j \mapsto \delta_{ij}} A$$

We can check that $\lambda \frac{\partial}{\partial t_j} = D_j \lambda$ for all j by checking on the generators. Hence the kernel $\ker \lambda$ is stable under $\frac{\partial}{\partial t_j}$. Therefore, $\ker \lambda$ has to contain some constant, which shows that $\ker \lambda = 0$. This proves that λ is an isomorphism, and hence the claim that H has order p^d.

Since G is connected, $\varphi_G^r = 0$ for some r. Since φ_G^r on G/H is 0, G/H is connected. Finally, the order of G is the order of H times the order of G/H. Induction hence completes the proof. □

Recall that if the order of G is invertible in the base, then G is étale. If R is a henselian local ring with perfect residue field, then there is another proof of the the proposition. Assume $R = k$ is a field. If k has characteristic p, the connected–étale sequence has $G^0 = 0$ if order is invertible in p. When k has characteristic 0, $G^0 \cong \operatorname{Spec}(k[t_1, \cdots, t_d])$ when $d = \dim I/I^2$, so $d = 0$.

2 p-divisible groups

The reference for this section are [Dem86] and [Tat67]. We assume throughout that the base ring R is a Henselian local noetherian ring.

2.1 Basic Definitions

Definition 2.1. *A p-divisible group of height h over R is an inductive system $G = \varinjlim G_v$ such that*

- *G_v is a finite flat R-groups of order $p^v h$.*
- *there is an exact sequence*
$$0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1},$$
i.e. $G_v = G_{v+1}[p^v]$.

Example 2.2. 1. *The constant p-divisible group is*
$$\underline{\mathbb{Q}_p/\mathbb{Z}_p} = \varinjlim \underline{\mathbb{Z}/p^v\mathbb{Z}}$$
with the obvious transfer maps. It is a p-divisible group of height 1.

2. *The p-power roots of unity is*
$$\mu_{p^\infty} = \varprojlim \mu_{p^v}$$
with the obvious transfer maps. It is a p-divisible group of height 1.

3. *If $\mathcal{A}$ is an abelian scheme over R,*
$$\mathcal{A}[p^\infty] = \varinjlim \mathcal{A}[p^v]$$
with the obvious transfer maps is a p-divisible group of height $2g$, where $g = \dim A$.

Definition 2.3. *A map of p-divisible groups $f : G \to H$ is a homomorphism if $f = (f_i)$ is compatible system of R-group homomorphism:*

$$\begin{array}{ccc} G_v & \xrightarrow{f_v} & H_V \\ \downarrow & & \downarrow \\ G_{v+1} & \xrightarrow{f_{v+1}} & H_{v+1} \end{array}$$

The kernel of f is $\ker(f) = \varinjlim \ker(f_v)$.

Remark 2.4. *The kernel of f might not be a p-divisible group.*

Example 2.5. *The map $[n]_G = ([n]_{G_v})$ is a homomorphism, called multiplication by n on G.*

We want to discuss Cartier duality for p-divisible groups. We first need a lemma

Lemma 2.6. *Let $G = (G_v)$ be a p-divisible group over R. Then for any $v, t \in \mathbb{Z}_{\geq 0}$ there exist*
$$i_{v,t} : G_v \hookrightarrow G_{v+t},$$
$$j_{v,t} : G_{v+t} \to G_t$$
such that

1. *$i_{v,t}$ induces $G_v = G_{v+t}[p^v]$,*

2. *the diagram*

$$\begin{array}{ccc} G_{v+t} & \xrightarrow{[p^v]} & G_{v+t} \\ & {\scriptstyle j_{v,t}} \searrow \quad \nearrow {\scriptstyle i_{v,t}} & \\ & G_t & \end{array}$$

   commutes.

3. *there is a short exact sequence*

$$0 \to G_v \xrightarrow{i_{v,t}} G_{v+t} \xrightarrow{j_{v,t}} G_r \to 0.$$

Proof. We have that $i_{v,t} = i_{v+t-1} \circ i_{v+t-2} \circ \cdots \circ i_v : G_v \hookrightarrow G_{v+t}$. To check (1), we see that

$$\begin{aligned} G_{v+t}[p^v] &= G_{v+1}[[p^{v+t-1}] \cap G_{v+t}[p^v] \\ &= G_{v+t-1} \cap G_{v+t}[p^v] \\ &= G_{v+t-1}][p^v]. \end{aligned}$$

To construct $j_{v,t}$, we first note that $[p^{v+t}]$ kills G_{v+t}. Hence $[p^v](G_{v+t})$ is killer by $[p^t]$. Hence

$$[p^v](G_{v+t}) \subseteq G_{v+t}[p^t] = G_t.$$

The composition defines a map $j_{v,t} : G_{v+t} \to G_t$ such that the diagram in (2) commutes.
Finally, it remains to check the surjectivity of $j_{t,v}$ to complete the proof of (3). We have that $\ker(j_{v,t}) = \ker[p^v] = G_v$. Hence $j_{v,t}$ induces a map

$$G_{v+t}/G_v \hookrightarrow G_t$$

between two groups of order $p^{v+t}/p^v = p^t$. It is hence an isomorphism, showing $j_{v,t}$ is surjective. □

Corollary 2.7. *The map $[p]$ on G is surjective as a map of fpqc schemes*

Proposition 2.8 (Cartier duality for p-divisible groups)**.** *Let $G = \varinjlim G_v$ be a p-divisible groups of height h over R.*

1. *The sequence*

$$G_{v+1} \xrightarrow{[p^v]} G_{v+1} \xrightarrow{j_v = j_{1,v}} G_v \to 0$$

   is exact.

2. *The injective limit $G^\vee = \varinjlim G_v^\vee$, the Cartier dual of G, is a p-divisible group of height h over R with transfer maps $j_v^\vee$.*

3. *There is a canonical isomorphism $G^{\vee\vee} \cong G$.*

Proof. We start with (1). We have a commutative diagram with an exact row:

$$\begin{array}{ccccccccc} & & & & G_1 & & & & \\ & & & {\scriptstyle j_{v,1}} \nearrow & & \searrow {\scriptstyle i_{1,v}} & & & \\ 0 \longrightarrow & G_V & \xrightarrow{i_v = i_{v,1}} & G_{v+1} & \xrightarrow{\quad [p^v] \quad} & G_{v+1} & \xrightarrow{j_{1,v} = j_v} & G_v & \longrightarrow 0 \end{array}$$

We have that $\ker(j_{1,v}) = G_1 = \text{im}([p^v]_{G_{v+1}})$. We hence get (1).

For (2), we dualize to get an exact sequence

$$0 \to G_v^\vee \xrightarrow{j_v^\vee} G_{v+1}^\vee \xrightarrow{p^v} G_{v+1}^\vee$$

by Cartier duality. Hence, $G_i^\vee = \varinjlim G_v^\vee$ is a p-divisible group. For part (3), it's similar to the proof of Cartier duality. □

Example 2.9. *We have that:*

1. $(\mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \mu_{p^\infty}$,
2. $\mathcal{A}[p^\infty]^\vee \cong \mathcal{A}^\vee[p^\infty]$.

Proposition 2.10 (Connected–étale sequence for p-adic groups)**.** *Let $G = \varinjlim G_v$ be a p-divisible group over R. Then there are p-divisible groups over R:*

$$G^0 = \varinjlim G_v^0,$$

$$G^{ét} = \varinjlim G_v^{ét}$$

such that

$$0 \to G^0 \to G \to G^{ét} \to 0.$$

Proof. We have a diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_v^0 & \longrightarrow & G_v & \longrightarrow & G_v^{\text{ét}} & \longrightarrow & 0 \\
 & & \vdots\, i_v^0 & & \downarrow\, i_v & & \vdots\, i_v^{\text{ét}} & & \\
0 & \longrightarrow & G_{v+1}^0 & \longrightarrow & G_{v+1} & \longrightarrow & G_{v+1}^{\text{ét}} & \longrightarrow & 0 \\
 & & \downarrow [p^v] & & \downarrow [p^v] & & \downarrow [p^v] & & \\
0 & \longrightarrow & G_{v+1}^0 & \longrightarrow & G_{v+1} & \longrightarrow & G_{v+1}^{\text{ét}} & \longrightarrow & 0
\end{array}$$

where the dotted maps are to be constructed. There is a unique $i_v^{\text{ét}}$ such that the top right square commutes. For exactness, we can pass to $\overline{k}$-points and see that it follows the middle column on $\overline{k}$-points. There is also a unique closed embedding i_v^0 such that the left top square commutes.
We want to show that $G_v^0 = G_{v+1}^0[p^v]$. Obviously, $G_v^0 \subseteq G_{v+1}^0[p^v]$. Also, $G_{v+1}^0[p^v] \subseteq G_v^0$ and $G_{v+1}^0[p^v] \subset G_{v+1}[p^v] = G_v$. Finally, $G_{v+1}^0[p^v](k) \subseteq G_{v+1}^0(k) = 0$. □

Definition 2.11. *Let $R = k$ be a perfect field of characteristic p. There is a Frobenius twist:*

$$G^{(p)} = \varinjlim G_v^{(p)}.$$

There is a Frobenius morphism $\varphi_G = (\varphi_{G_v})$ and a Verschiebung morphism $\psi_G = (\psi_{G_v})$.

Proposition 2.12. *If G is a p-divisible group of height h,*

1. *$G^{(p)}$ is a p-divisible group of height h,*
2. *φ_G and ψ_G are homomorphisms,*
3. *$\psi_G \circ \varphi_G = [p]_G$,*
4. *$\varphi_G \circ \psi_G = [p]_{G^{(p)}}$.*

Definition 2.13. *Let $R = k$ be a field. The Tate module of G is*

$$T_p(G) = \varprojlim G_v(\overline{k}),$$

where the transfer maps are given by $j_v : G_{v+1} \to G_v$.

Proposition 2.14. *Let $R = k$ be a field of characteristic not equal to p. Then there is an equivalence:*

$$\{p\text{-divisible groups over } k\} \Longrightarrow \{\text{ finite free } \mathbb{Z}_p\text{-modules with continuous } \Gamma_K\text{-action}\},$$

$$G \mapsto T_p(G).$$

Proof. Use the corresponding equivalence for finite flat k-groups and the fact that groups with invertible orders are étale. □

3 Serre–Tate equivalence for connected p-divisible groups.

A key correspondence for p-divisible groups is the Serre–Tate equivalence:

$\{$connected p-divisible group over $R\} \leftrightarrow \{$formal group laws over $R\} \leftrightarrow \{p$-divisible formal Lie groups$\}$.

Let R be a complete local noetherian ring, with residue characteristic p.

Definition 3.1. *Let $G = \varinjlim G_v$ be a p-divisible group over R. We say that G is:*

- *connected if each G_v is connected,*
- *étale if G_v is étale.*

Example 3.2. *1. The p-divisible group μ_{p^∞} is connected.*

2. The p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is étale.

Definition 3.3. *Let $\mathcal{A} = R[\![t_1, \cdots, t_d]\!]$. Then define*

$$\mathcal{A}\widehat{\otimes}\mathcal{A} = R[\![t_1, \cdots, t_d, u_1, \cdots, u_d]\!].$$

We will also write $T = (t_1, \cdots, t_d), U = (u_1, \cdots, u_D)$ for the variables.
A formal group law of dimension d over R is a (continuous) map $\mu : \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\Phi(T, U) = (\Phi_i(T, U))$ for each $\Phi_i(T, V)$ a power series of $2d$ variables and

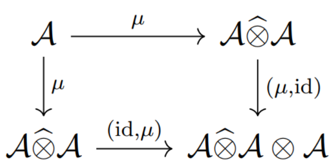
$$\Phi_i(T, V) = \mu(t_i)$$

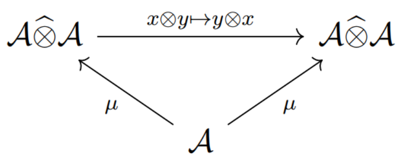
satisfying the following properties:

1. *associativitiy: $\Phi(T, \Phi(V, V)) = \Phi(\Phi(T, V), V)$,*
2. *unit section: $\Phi(T, 0_d) = \Phi(0_d, T) = T$,*
3. *commutativity: $\Phi(T, V) = \Phi(V, T)$.*

Lemma 3.4. *If μ is a formal group law over R, then*

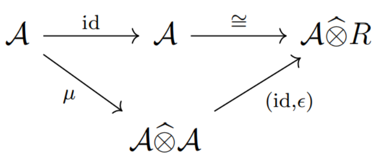
1. *the diagrams*





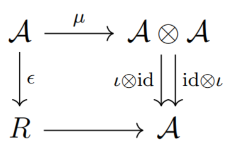
commute,

2. *the map $\epsilon : \mathcal{A} \to R$ given by $t_i \mapsto 0$ makes the diagram*



and a symmetric diagram commute,

3. *there is a continuous map $\iota : \mathcal{A} \to \mathcal{A}$ such that*



commutes.

Proof. Parts (1) and (2) are clear. For (3), we need to define $I_i(T) = \iota(t_i), I(T) = I_i(T)$ such that

$$\Phi(I(T), T) = 0 = \Phi(T, I(T)).$$

We want $P_j(T)$: a family of polynomials of degree j such that $I(T) = \lim P_j(T)$, i.e.

1. $P_j(T) = P_{j-1}(T)$ mod degree j,
2. $\Phi(P_j(T), T) = 0$ mod degree $j + 1$.

Since $\Phi(T, U) = T + U$ mod degree 2, we may take $P_1(T) = -T$. We define $P_j(T)$ by recursion on j. We have that

$$\Phi(P_j(T), T) = \Delta_j(T) \text{ mod degree } j + 2,$$

where $\Delta_j(T)$ is a homogeneous polynomial of degree $j + 1$. Define

$$P_{j+1}(T) = P_j(T) + \Delta_j(T).$$

Then (i) is clearly satisfied. For (ii), we note that

$$\Phi(P_{j+1}(T), T) = \Phi(P_j(T) + \Delta_j(T), T) \cong \Phi(P_j(T), T) + \Delta_j(T) \equiv 0 \text{ mod degree } j + 2.$$

This proves (3). □

Remark 3.5 (Formal schemes and groups)**.** *A formal scheme is a scheme together with an infinitesimal neighborhood.*
If A is a ring, we define $\mathrm{Spec}(A)$ *as the set of prime ideals.*
If A is a topological ring, we define Spf(A), the formal spectrum, as the set of open prime ideals of A. Formal groups are group objects in the category of formal schemes. The lemma says that any formal group law over R defines a formal group structure on Spf(A), written G_μ.

Example 3.6. *The multiplicative formal group law is*

$$\mu_{\widehat{\mathbb{G}_m}} : R[\![t]\!] \to R[\![t, u]\!],$$

$$t \mapsto (1 + t)(1 + u) - 1.$$

Definition 3.7. *Let μ, ν be formal group laws of dimension d over R. A continuous map $\gamma A \to A$ is a homomorphism from μ to ν if the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\nu} & A \widehat{\otimes} A \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma\otimes\gamma} \\ A & \xrightarrow{\mu} & A \widehat{\otimes} A \end{array}$$

commutes.

Lemma 3.8. *A continuous map $\gamma : A \to A$ given by $\Xi(T) = (\Xi_i(T))$ where $\Xi_i(T) = \gamma(t_i)$ if a homomorphism if and only if, writing $\Phi(T, V)$ and $\Psi(T, V)$ for the functions associated to μ and ν, we have that*

$$\Psi(\Xi(T), \Xi(V)) = \Xi(\Phi(T, V)).$$

Example 3.9. *The multiplication by n map $[n]_\mu$ on μ is a homomorphism.*

Definition 3.10.
1. *The ideal $I = (t_1, \cdots, t_d) = \ker \epsilon$ is the augmentation ideal of μ.*
2. *A formal group law μ is p-divisible if $[p]_\mu$ is finite flat in the sense that A is a free module of finite rank over itself.*

Remark 3.11. *A formal group law μ is p-divisible if and only if $[p]$ on G_μ is surjective with finite kernel.*

Proposition 3.12. *Let μ be a p-divisible formal group law of dimension d over R. Define*

$$A_v = \mathcal{A}/([p^v]_\mu(\mathcal{I})),$$

$$A[p^v] = \operatorname{Spec}(A_v).$$

Then

1. *each $\mu[p^\vee]$ is a connected finite flat R-group,*
2. *$\mu[p^\infty] = \varinjlim \mu[p^\vee]$ is a connected p-divisible group over R.*

Proof. We may write

$$\begin{aligned} A_v &= \mathcal{A}/[p^\vee]_\mu(\mathcal{I}) \\ &= (\mathcal{A}/\mathcal{I}) \otimes_{\mathcal{A},[p^v]} \mathcal{A} \\ &= R \otimes_{\mathcal{A},[p^v]} \mathcal{A}. \end{aligned}$$

Then $1 \otimes \mu, 1 \otimes \epsilon, 1 \otimes \iota$ define comultiplication, counit, and coinverse on A_V.
Let r be the rank of $\mathcal{A}$ over $[p](\mathcal{A})$. Then r^v is the rank of A over $[p^v](\mathcal{A})$. Hence $\operatorname{Spec}(A_v)$ is a finite flat R-group scheme of order r^v.
Since R is complete, $\mathcal{A}$ is also a lcoal ring. Hence each A_v is a local ring, showing that $\operatorname{Spec}(A_v)$ is connected over R. Since $\operatorname{Spec}(A_1)$ has order $p^h = r$, and $\operatorname{Spec}(A_v)$ has order p^{hv}. This completes the proof of (1).
For (2), we need to check that $\mu[p^v]$ is the p^v-torsion of $\mu[p^{v+1}]$. The natural surjective map

$$A_v = \mathcal{A}/[p^v](\mathcal{I}) \twoheadrightarrow [p]\mathcal{A}/[p^{v+1}](\mathcal{I})$$

is an isomorphism as it is an R-linear map between R-modules of the same rank. We hence have a surjection

$$A_{v+1} = \mathcal{A}/[p^{v+1}](\mathcal{I}) \twoheadrightarrow [p]A/[p^{v+1}](\mathcal{I}) \cong A_v$$

induced by $[p]$, and hence $[p^v]$ will be 0. □

Remark 3.13. *We have that $G_\mu[p^v] = \operatorname{Spec}(A_v)$.*

Theorem 3.14 (Serre-Tate equivalence)**.** *There functor*

$$\{p\text{-divisible formal group laws over } R\} \to \{\text{connected } p\text{-divisible groups over } R\}$$

$$\mu \mapsto \mu[p^\infty]$$

is an equivalence of categories.

The map above is really the following. We have a formal group scheme G_μ associated to μ. Then the connected p-divisible group over R associated to μ is

$$\varinjlim G_v \cong \varinjlim G_\mu[p^v],$$

where we recall that

$$G_v = \operatorname{Spec}(\mathcal{A}/[p^v](\mathcal{I})).$$

Remark 3.15. *Local class field theory can be stated in terms of Lubin–Tate formal group laws. Local Langlands for GL_1 is local class field theory. It can hence be stated in terms of certain p-divisible groups.*
For GL_n, Michael Harris and Richard Taylor proved the local Langlands correspondence via moduli spaces of p-divisible groups: Rapoport–Zink spaces and local Shimura varieties.

We now work towards the proof of the Serre–Tate equivalence. The following proposition shows the essential surjectivity over k in the Serre–Tate equivalence.

Proposition 3.16. *Let $G = \varinjlim G_v$ be a connected p-divisible group over R, where $G_v = \operatorname{Spec}(A_v)$. Then*

$$\varprojlim A_v \otimes k \cong k[\![t_1, \cdots, t_d]\!].$$

Proof. Let $\overline{G} = G \times_R k$. Define $H_v = \ker(\varphi^v)$ and note that $H^v \subseteq \ker([p^v]) = \overline{G_v}$. Since

$$\varphi^v \circ \varphi^v = [p^v],$$

writing $H_v = \mathrm{Spec}(B_v)$ and we have $A_v \otimes k \twoheadrightarrow B_v$.
We have that $\overline{G_v}$ is a connected finite flat k-group. Hence $\varphi^w = 0$ on $\overline{G_v}$, so $G_v \subset H_w$ showing that $B_w \twoheadrightarrow A_v \otimes k$. Hence

$$\varprojlim A_v \otimes k \cong \varprojlim B_v.$$

Let J_v be the augmentation ideal of H_v and $J = \varprojlim J_v$. Then $B_v/J_v \cong k$. Let $y_1, \cdots, y_d \in J$ lift a basis of J_1/J_1^2. We have a commutative diagram:

$$\begin{array}{ccc} k = (B_v/J_v) \otimes_{k,\sigma} k & \longrightarrow & B_1 \\ \uparrow & & \uparrow \\ B_v^{(p)} = B_v \otimes_{k,\sigma} k & \xrightarrow{x \otimes c \to cx^p} & B_v \end{array}$$

so

$$B_1 \cong B_v/J_v^{(p)},$$

where $J_v^{(p)}$ is the ideal generated by p-powers of elements in J. Since $J_1/J_1^2 \cong J_v/J_v^2$, the images of $y_1, \cdots, y_d$ generate J_v/J_v^2. By Nakayma's Lemma, they generate J_v. We hence have a map

$$k[t_1, \cdots, t_d] \twoheadrightarrow B_v.$$

We hence have

$$k[t_1, \cdots, t_d]/(t_1^{p^v}, \cdots, t_d^{p^v}) \twoheadrightarrow B_v,$$

since $H_v = \ker(\varphi^v)$. We want to show this is an isomorphism.
We proceed by induction on v. When $v = 1$, we checked this in the proof of Proposition 1.8. For the induction step, we argue on ranks. We want to show that p^{vd} is the order of H_v. For that, we observe that the sequence

$$0 \to H_1 \to H_{v+1} \xrightarrow{\varphi} H_v^{(p)} \to 0$$

is exact. Since $H_1 = \ker(\varphi)$, we just need to check that φ is surjective. Recall that $[p]$ is surjective by Corollary 2.7. We know that $\varphi \circ \psi = [p]$, so φ is surjective. Recall that $H_{v+1} = \ker(\varphi^{v+1}), so \varphi(H_{v+1}) \subseteq \ker(\varphi^\vee_{G^{(p)}})$, and the preimage of $H_v^{(p)}$ is $\ker(\varphi^\vee_{G^{(p)}})$.
This shows that the order of H_{v+1} is $p^d \cdot p^{vd} = p^{d(v+1)}$, completing the proof. ☐

Lemma 3.17. *Let μ be a p-divisible formal group law R. Letting*

$$A_v = \mathcal{A}/[p^v](\mathcal{I}),$$

we have that

$$\mathcal{A} \cong \varprojlim A_v.$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Then $\mathfrak{M} = \mathfrak{m}\mathcal{A} + \mathcal{I}$ is a maximal ideal of $\mathcal{A}$. For each v, i, we have that

$$[p^v](\mathcal{I}) + \mathfrak{m}^i \mathcal{A} \supseteq \mathfrak{M}^w$$

for some w, since

$$\mathcal{A}/([p^v](\mathcal{I}) + \mathfrak{m}^i \mathcal{A}) = A_v/\mathfrak{m}^i A_v,$$

which is local Artinian. ☐

Moreover, $[p](\mathcal{I}) \subset p\mathcal{I} + \mathcal{I}^2$, because $[n]$ acts as multiplication by n on $\mathcal{I}/\mathcal{I}^2$. Alternatively, recall that $\Phi(T, U) = T + U + (\deg \geq 2)$. This shows that

$$[p^v](\mathcal{I}) + \mathfrak{m}^i \mathcal{A} \subseteq \mathfrak{M}^{w'}$$

for some w'.

Altogether, we see that

$$\begin{aligned}\mathcal{A} &\cong \varprojlim \mathcal{A}/\mathfrak{M}^w \\ &= \varprojlim_{v,i} \mathcal{A}/([p^v](\mathcal{I}) + \mathfrak{m}^i\mathcal{A}) \\ &= \varprojlim_{v,i} A_v/\mathfrak{m}^i A_v \\ &\cong \varprojlim_{v} A_v\end{aligned}$$

where the last congruence follows from the fact that A_v is $\mathfrak{m}$-adically complete. This completes the proof.

Proof of Serre-Tate Equivalence. We first check that the functor is fully faithful. Let μ, ν be p-divisible formal group laws over R. Then for $B_v = \mathcal{A}/[p^v]_\nu(\mathcal{I})$:

$$\begin{aligned}\mathrm{Hom}(\mu,\nu) &= \mathrm{Hom}_{\nu,\mu}(\mathcal{A},\mathcal{A}) \\ &= \mathrm{Hom}_{\nu,\mu}(\varprojlim B_v, \varprojlim A_v) && \text{by lemma 3.17} \\ &= \varprojlim \mathrm{Hom}_{\nu_v,\mu_v}(B_v, A_v) \\ &= \varinjlim \mathrm{Hom}_{grp}(\mu[p^v], \nu[p^v]) \\ &= \mathrm{Hom}(\mu[p^\infty], \nu[p^\infty]).\end{aligned}$$

For essential surjectivity, consider $G = \varinjlim G_v$ be a connected p-divisible group. Let

$$\overline{G} = G \times_R k,$$

and $G_v = \mathrm{Spec}(A_v)$, then by proposition 3.16, we have

$$k[\![t_1, t_2, \cdots t_d]\!] \cong \varprojlim A_v \otimes k.$$

We want to lift to $f : \mathcal{A} \to \varprojlim A_v$. We hence need lifts $f_v : \mathcal{A} \to A_v$, which lifts the above isomorphism, such that

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{f_{v+1}} & A_{v+1} & \longrightarrow & A_{v+1}\otimes k \\ & {\scriptstyle f_v}\searrow & \downarrow{\scriptstyle p_v} & & \downarrow \\ & & A_v & \longrightarrow & A_v \otimes k. \end{array}$$

Let f_1 be any lift over $k[\![t_1, \cdots, t_d]\!] \to A_1 \otimes k$. We define f_v by recursion on v. Choose $y_1, \cdots, y_d \in A_{v+1}$ which lift images of $t_1, \cdots, t_d$ under

$$k[\![t_1, \cdots t_d]\!] \to A_{v+1} \otimes k.$$

Then $p_v(y_1), \cdots p_v(y_d)$ must lift the images of $t_1, \cdots t_d$ after the map

$$k[\![t_1, \cdots t_d]\!] \to A_v \otimes k.$$

We know that $f_v(t_1), \cdots, f_v(t_d)$ also lift the images of $t_1, \cdots, t_d$ under this map. Then $f_v(t_i) - p_v(y_i) \in \mathfrak{m}A_v$, so there exist $z_i \in \mathfrak{m}A_{v+1}$ such that

$$p_v(z_v) = f_v(t_i) - p_v(y_i).$$

Defining f_{v+1} by $f_{v+1}(t_i) = y_i + z_i$ gives the desired lift. We want to show that the resulting map

$$f : \mathcal{A} \to \varprojlim A_v$$

is an isomorphism. Surjectivity is clear by Nakayama's Lemma. We want to show that $\ker(f) = 0$. We know that $\ker(f) \otimes_R k = 0$, i.e. $\mathfrak{m}\ker(f) = \ker(f)$. We now note that

$$\mathfrak{M}\ker(f) = (\mathfrak{m}\mathcal{A} + \mathcal{I})(\ker f) = \ker f.$$

so f is injective by Nakayama's Lemma.
We have an isomorphism

$$f : \mathcal{A} \to \varprojlim A_v.$$

To prove essential surjectivity, We define $G = \varinjlim G_v$ for $G_v = \mathrm{Spec}(A_v)$. Then μ_v is a comultiplication on G_v, and $\mu = \varprojlim \mu_v$ defines a formal group law over R such that $\mu[p^v] = G_v$.
We just need to check that G is p-divisible. We omit the details of this; roughly, ones uses that the map $j_{v,t} : G_{v,t} \twoheadrightarrow G_t$ induces an injection $A_t \hookrightarrow A_{v+t}$. For details, check Serin Hong's notes. ☐

Definition 3.18. *For a p-divisible group $G = \varinjlim G_v$ over R,*

$$\dim(G) = \textit{dimension of the formal group la associated to } G^0$$

via the Serre-Tate equivalence 3.14.

In the course of the proof of Theorem 3.14, we showed the following result:

Corollary 3.19. *Let $\overline{G} = G \times_R k$. Then $\ker(\varphi_{\overline{G}})$ has order $p^{\dim(G)}$.*

Example 3.20. *Recall that $\mu_{\widehat{G_m}}(t,u) = (1+t)(1+u) - 1$. Then $[p^v](t) = (1+t)^{p^v} - 1$, so*

$$\mu_{\widehat{G_m}}[p^\infty] = \mu_{p^\infty}.$$

Theorem 3.21. *Let G be a p-divisible group over R. Then*

$$ht(G) = \dim(G) + \dim(G^\vee).$$

Proof. By passing to the residue field, we may assume that $R = k$ is a perfect field of characteristic p. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\varphi) & \longrightarrow & G & \xrightarrow{\varphi} & G^{(p)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle [p]} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & G & \xrightarrow{\text{id}} & G & \longrightarrow & 0 \end{array}$$

is commutative with exact rows, since φ is surjective, because $\varphi \circ \psi = [p]_{G^{(p)}}$ and $\ker(\varphi)$ is killed by $[p]$ because $\psi \circ \varphi = [p]$. Snake Lemma then gives a short exact sequence

$$0 \to \ker \varphi \to \ker([p]) \to \ker \psi \to 0.$$

Since $\ker(\varphi)$ has order $p^{\dim(G)}$ and $\ker([p]) = G_1$ has order $p^{ht(G)}$, and $\psi = \varphi^\vee_{G^\vee}$ implies that $\ker(\psi)$ has order $p^{\dim(G^\vee)}$, we are done by multiplicativity of orders in short exact sequences. ☐

Corollary 3.22. *Let G be a p-divisible group over R with residue field $k = \overline{k}$ of height 1. Then G is isomorphic to μ_{p^∞} or $\mathbb{Q}_p/\mathbb{Z}_p$.*

Proof. By theorem 3.21, we know that $\dim G = 0$ or $\dim G^\vee = 0$. If $\dim G = 0$, G is étale, so $G \cong \mathbb{Q}_p/\mathbb{Z}_p$. Otherwise, $\dim G^\vee = 0$, so $G^\vee \cong \mathbb{Q}_p/\mathbb{Z}_p$, so $G \cong \mu_{p^\infty}$. ☐

One can also prove this result using Dieudonné theory, which we will soon explain.

Example 3.23. *Let E be an ordinary elliptic curve over $\mathbb{F}_p$. Then there is a short exact sequence*

$$0 \to E[p^\infty]^0 \to E[p^\infty] \to E[p^\infty]^{\acute{e}t} \to 0.$$

Since $E[p]^0$ and $E[p]^{\acute{e}t}$ are both non-trivial, so are $E[p^\infty]$ and $E[p^\infty]^{\acute{e}t}$. Finally, $E[p^\infty]$ is of height 2, so Corollary 3.22 shows that

$$E[p^\infty]^0 = \mu_{p^\infty}, \quad E[p^\infty]^{\acute{e}t} = \underline{\mathbb{Q}_p/\mathbb{Z}_p}.$$

The short exact sequence splits, because it splits at each finite level. Hence

$$E[p^\infty] = \mu_{p^\infty} \times \underline{\mathbb{Q}_p/\mathbb{Z}_p}.$$

Remark 3.24. *We discuss Serre–Tate deformation theory for ordinary elliptic curves.*
In general, Serre–Tate deformation theory says that the deformations of an abelian variety A/k are equivalent to the deformations of $A[p^\infty]$ (i.e. p-divisible groups G/R such that $G \times_R k \cong A[p^\infty]$).

Therefore a deformation of an elliptic curve E over $k = \overline{k}$ corresponds to a deformation of $E[p^\infty]$. The deformation space of $E[p^\infty]$ is

$$\underline{\text{Ext}}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}),$$

since if G is a deformation over R, the connected–étale sequence

$$0 \to G^0 \to G \to G^{\acute{e}t} \to 0$$

and $G^0 = \mu_{p^\infty}$ and $G^{\acute{e}t} = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$.
We also have a short exact sequence

$$0 \to \underline{\mathbb{Z}_p} \to \underline{\mathbb{Q}_p} \to \underline{\mathbb{Q}_p/\mathbb{Z}_p} \to 0$$

The long exact sequence after applying $Ext(-, \mu_{p^\infty})$ gives

$$\underline{\text{Ext}}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}) \cong \underline{\text{Hom}}(\underline{\mathbb{Z}_p}, \mu_{p^\infty}).$$

Therefore, the deformation space has the structure of a formal torus of dimension 1, given by $\mu_{\widehat{G_m}}$.

4 Dieudonné–Manin classification

Let k be a perfect field of characteristic p. Let σ be the Frobenius automorphism over k.

Definition 4.1. *We write $W(k)$ for the ring of Witt vectors over k. We write $K_0(k)$ for the fraction field of $W(k)$. The Frobenius $\sigma_{W(k)}$ on $W(k)$ is*

$$\sigma\left(\sum_{n\geq 0}\tau(x_n)p^n\right) = \sum_{n\geq 0}\tau(x_n^p)p^n$$

where $\tau : k \to W(k)$ is the Teichmuller lift. Finally, $\sigma_{K_0}(k)$ is the unique field of automorphism on $K_0(k)$ extending $\sigma_{W(k)}$.

Example 4.2. *Let $k = \mathbb{F}_q$ and ζ_{q-1} be a primitive $(q-1)$st root of unity. Then*

$$W(k) = \mathbb{Z}_p[\zeta_{q-1},\ K_0(k) = \mathbb{Q}_p[\zeta_{q-1}]$$

and σ acts on $W(k)$ by

$$\sigma(\zeta_{q-1}) = \zeta_{q-1}^p,$$

and trivially on $\mathbb{Z}_p$.

Definition 4.3. *A Dieudonné module over k is a pair (M, φ) where*

1. *M is a finite free module over $W(k)$,*
2. *$\varphi : M \to M$ is an additive map such that:*
   (a) *φ is σ-linear, i.e. $\varphi(am) = \sigma(a)\varphi(m)$ for all $a \in W(k), m \in M$,*
   (b) *$\varphi(M) \supseteq pM$.*

Theorem 4.4 (Dieudonné)**.** *There is an anti-equivalence:*

$$\mathbb{D} : \{p\text{-divisible groups over } k\} \to \{\text{Dieudonné modules over } k\}$$

such that

1. *$rk(\mathbb{D}(G)) = ht(G)$,*
2. *G is étale if and only if $\varphi_{\mathbb{D}(G)}$ is an isomorphism.*
3. *G is connected if and only if $\varphi_{\mathbb{D}(G)}$ is topologically nilpotent,*
4. *$[p]_G$ induces multiplication by p on $\mathbb{D}(G)$.*

We shall omit a proof here. But the interested reader could read [Dem86].

Remark 4.5. *There is a notion of duality for Dieudonn ´e modules, compatible with Cartier duality.*