p-adic Hodge Theory (Spring 2023): Week 2

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1 From last time: the interplay via representation theory

The Grothendieck mysterious functor, which we have yet to give a complete description, is an example of various functors that link the arithmetic side and the geometric side of p-adic Hodge theory. Such functors provide vital means for studying p-adic Hodge theory via the interplay between the arithmetic and geometric perspectives.

Here we describe a general formalism due to Fontaine for constructing functors that connect the arithmetic and geometric sides of p-adic Hodge theory. Let $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ denote the category of p-adic representations of Γ_K for a p-adic field K. For a p-adic period ring B, such as B_{HT}, B_{dR} or B_{cris} as introduced in the preceding subsection, we define

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \quad \text{for each } V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$$

We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-*admissible* if the natural morphism

$$\alpha_V : D_B(V) \otimes_{B^{\Gamma_K}} B \to V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism. Let $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ be the full subcategory of B-admissible representations. Then D_B defines a functor from $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ to the category of finite dimensional vector spaces over B^{Γ_K} with some additional structures. Here the additional structures that we consider for the target category reflect the structure of the ring B, as indicated by the following examples:

1. The target category of $D_{B_{HT}}$ is the category of finite dimensional graded K-spaces, reflecting the graded algebra structure on B_{HT}.
2. The target category of $D_{B_{dR}}$ is the category of finite dimensional filtered K-spaces, reflecting the filtration on B_{dR}. The target category of $D_{B_{cris}}$ is the category of finite dimensional filtered K_0-spaces[1] with a Frobenius-semilinear endomorphism, reflecting the filtration and the Frobenius action on B_{cris}.

In particular, we have a complete description of the Grothendieck mysterious functor given by $D_{B_{cris}}$. We also obtain its fully faithfulness from the following fundamental result:

Theorem 1.1 (Fontaine)**.** *The functors $D_{B_{HT}}$, $D_{B_{dR}}$, and $D_{B_{cris}}$ are all exact and faithful. Moreover, the functor $D_{B_{cris}}$ is fully faithful.*

Remark 1. *We will see during week 6 that the first statement of Theorem 1.1 is (almost) a formal consequence of some algebraic properties shared by B_{HT}, B_{dR}, B_{cris}.*

Note that, for each $B = B_{HT}, B_{dR}, B_{cris}$, the definition of B-admissibility is motivated by the corresponding comparison theorem from the preceding subsection, while the target category of the functor D_B consists of semilinear algebraic objects that arise in the arithmetic side of p-adic Hodge theory. In other words, the functor D_B relates a certain class of "geometric" p-adic representations to a class of semilinear algebraic objects that carry some arithmetic information. Hence we can consider Fontaine's formalism as a general framework for connecting the following themes:

1. Study of the geometry of a proper smooth variety over a p-adic field via the Galois action on the étale cohomology groups.

[1] Recall: Let k be the residue field of $\mathcal{O}_K$. Denote by $W(k)$ the ring of Witt vectors over k, and by K_0 its fraction field.

2. Construction of a dictionary that relates certain p-adic representations to various semilinear algebraic objects.

In fact, this tidy connection provided by Fontaine's formalism forms the backbone of classical p-adic Hodge theory.

This week: Introduction: A first glimpse of Fargues Fountaine curve

2 Definitions and Key Features

There are two ways to describing the Fargues–Fontaine curve, the schematic curve and the adic curve. We will only describe the schematic curve, since we do not have the necessary language to talk about adic spaces. Fortunately, there is a GAGA type theorem (Algebraic Geometry-Analytic Geometry, not lady gaga), giving an equivalence between these two approaches.

For simplicity, we work with $K = \mathbb{Q}_p$. Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$, and let $F = \widehat{\overline{\mathbb{F}_p((u))}}$ (Here u is a variable: we want F to be a perfectoid field over $\mathbb{F}_p$).
Recall Fontaine's ring B_{cris} with Frobenius action φ. There is a ring B^+_{cris} such that

1. B^+_{cris} is stable under φ, and $(B^+_{cris})^{\varphi=1} = \mathbb{Q}_p$
2.) there exists $t \in B^+_{cris}$ such that $B^+_{cris}[\frac{1}{t}] = B_{cris}$ and $\varphi(t) = pt$.

Definition 2.1. *The Fargues–Fontaine curve associated to* $(\mathbb{Q}_p, F)$ *is*

$$X = \text{Proj}\left(\bigoplus_{n\geq 0}(B^+_{cris})^{\varphi=p^n}\right)$$

(For more details on Proj construction see here) We note as we are able to construct "structure sheaf", this makes X into a scheme.

Remark 2. *The Fagues-Fontaine Curve X is*

1. *a $\mathbb{Q}_p$-scheme.*
2. *not of finite type over $\mathbb{Q}_p$, hence not projective.*

Punchline: The Fargues–Fontaine curve is the p-adic analogue of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$.

Theorem 2.2 (Fargues-Fontaine, Kedlaya)**.** *The curve X satisfies the following properties:*

1. *it is Noetherian, connected, regular of dimension 1 over $\mathbb{Q}_p$,*
2. *it is the union of two spectra of Dedekind domains.*
3. *it is complete in the sense that for all $f \in K(X)$, $div(f)$ has degree 0,*
4. $\text{Pic}(X) \cong \mathbb{Z}$.

In fact, X is an affine scheme of a PID together with a point at ∞. There exist closed points $x \in X$ such that

$$X \setminus \{x\} \cong \text{Spec}(B_e)$$

$$\widehat{\mathcal{O}_{X,x}} \cong B^+_{dR}$$

where

$$B_e = B^{\varphi=1}_{cris},$$

$$B^+_{dR} = \textit{valuation ring of } B_{dR}.$$

3 Relation to theory of perfectoid spaces

Definition 3.1. *Let C be a field which is complete, non-archimedean, residue characteristic p.*

1. *It s a perfectoid field if*
   (a) *the valuation ring is non-discrete*
   (b) *the p-power map is surjective on $\mathcal{O}_C/p$.*
2. *The tilt of C is defined as*

$$C^\flat = \varprojlim_{x \mapsto x^p} C$$

with

$$(a \cdot b)_n = a_n \cdot b_n$$
$$(a+b)_n = \lim_{n\to\infty}(a_{n+m}b_{n+m})^{p^m}$$
$$|a|^\flat = |a_0.|$$

Remark 3. *For any C, $C^\flat$ is a perfectoid field of characteristic p.*

Example 3.2. *The field $\mathbb{C}_p$ is perfectoid of characteristic 0 with $\mathbb{C}_p^\flat \cong F$.*

Remark 4. *Scholze extended the de Rham Comparison Theorem (cf. last week) to rigid analytic varieties using the theory of perfectoid spaces.*

Theorem 3.3 (Tilting equivalence)**.** *Suppose C is a perfectoid field.*

1. *Every finite extension of C is a perfectoid field.*
2. *There is a bijection*

$$\{\textit{finite extension of } C\} \Longleftrightarrow \{\textit{finite extension of } C^\flat\}$$
$$L \mapsto L^\flat$$

3. *The above bijection induces an isomorphism $\Gamma_C \cong \Gamma_{C^\flat}$.*

This allows to translate problems in characteristic 0 to problems in characteristic p.

Question: Can we parametrizer a way of untilting?

Definition 3.4. *An untilt of F is a pair (C, ι), where C is a perfectoid field of characteristic 0, and $\iota : C^\flat \cong F$.*

Let φ_F be the Frobenius automorphism on F. It acts on the set of untilts of F by

$$\varphi_F \circ (C, \iota) = (C, \varphi_F \circ \iota).$$

Theorem 3.5 (Fargues-Fontaine)**.**
1. *For any closed point $x \in X$, $k(x)$ is a perfectoid field of characteristic* 0 *with $k(x)^\flat \cong F$.*
2. *There is a bijection*

$$\{\textit{closed points on } X\} \Longleftrightarrow \{\varphi_F\textit{-orbits of untilts}\}$$

induced by $x \mapsto k(x)$.

Remark 5. *This theorem is one of the main motivations for the theory of diamonds. Just as*

Algebraic space = scheme/étale equivalence relation

one should think that

Diamond = Perfectoid Space/pro-étale equivalence relation

4 Geometrization of p-adic representations

Definition 4.1. *Fix a closed point $\infty \in X$*

1. *A vector bundle on X is a locally free $\mathcal{O}_X$-module of finite rank.*
2. *A modification of vector bundles on X is $(\mathcal{E}, \mathcal{F}, i)$ where*
   - *$\mathcal{E}, \mathcal{F}$ are vector bundles on X*
   - *$i : \mathcal{E}|_{X \setminus \infty} \cong \mathcal{F}|_{X \setminus \infty}$.*

Remark 6. *There is a complete classification of vector bundles on X. We will see this later in our seminar. Roughly, it is analogous the fact that any vector bundle on $\mathbb{P}^1$ is isomorphic to $\oplus \mathcal{O}(\lambda)$. .*

Theorem 4.2 (Fargues-Fontaine)**.** *There is a functorial commutative diagram:*

$$\begin{array}{ccc} \{\text{isocrystals over } \overline{\mathbb{F}_p}\} & \xrightarrow{\sim} & \{\text{vector bundles on } X\} \\ \uparrow & & \uparrow \\ \{\text{filtered isocrystals over } \overline{\mathbb{F}_p}\} & \longrightarrow & \{\text{modifications of vector bundles on } X\} \end{array}$$

where the vertical arrows are forgetful functors. The top horizontal arrow is a bijection, but not an equivalence of categories.

Recall that there is a functor

$$D_{B_{\text{cris}}} : \{B_{cris}\text{-admissible representations over } \mathbb{Q}_p\} \to \{\text{filtered isocrystals over } \mathbb{F}_p\}$$

which is fully faithful.

So from theorem 4.2 we actually obtain the following:

$$\{B_{cris}\text{-admissible representations over } \mathbb{Q}_p\} \to \{\text{modifications of vector bundles on } X\}$$

Hence we can study B_{cris}-admissible p-adic Galois representations by purely geometric objects, namely modifications of vector bundles on X.

Question: What is the essential image of this functor?

Theorem 4.3 (Colmez-Fontaine)**.** *Given $N^0 = (N, Fil^\bullet(N))$ over $\mathbb{F}_p$, define $\overline{N^0} = (\overline{N}, Fil^\bullet(\overline{N}))$ over $\overline{\mathbb{F}_p}$. Via theorem 4.2, we obtain a modification of vector bundles $(\mathcal{E}(\overline{N}), \mathcal{F}(\overline{N}), i(\overline{N}))$.*

Then N^0 is in the essential image of $D_{B_{cris}}$ if and only if $\mathcal{F}(\overline{N})$ is trivial (i.e. $\mathcal{F}(\overline{N}) \cong \mathcal{O}_X^{\oplus n}$).

Remark 7. *Let V_{cris} be the quasi-inverse of $D_{B_{cris}}$. Then*

$$V_{cris}(\overline{N}) = H^0(X, \mathcal{F}(\overline{N})).$$

Implications:

- B_{cris}-admissibility is a "geometric" property
- B_{cris}-admissibility is insensitive to replacing the residue field $\mathbb{F}_p$ by $\overline{\mathbb{F}_p}$, which amounts to replacing the ground field $\mathbb{Q}_p$ by $\widehat{\mathbb{Q}_p^{un}}$.

These two implications are closely related since base change of the ground field $\mathbb{Q}_p$ to $\widehat{\mathbb{Q}_p^{un}}$ can be regarded as "passing to the geometry" via the bottom map in theorem 4.2.

Note: The Fargues-Fontaine Curve also provides a way to geometric sutdy ℓ-adic galois representations. And interested reader should read more about *geometric local langlands.*

5 Foundations of p-adic Hodge Theory

The goal of this chapter is to discuss:

1. finite flat group schemes,
2. p-divisible groups.

In particular, we will try to cover the main results of Tate's p-divisible groups [Tat67].

5.1 Finite flat group schemes

The main reference for this chapter is Tate's finite flat group schemes [Tat97].

5.1.1 Basic definition and properties

Definition 5.1. *Let S be a base scheme. An S-scheme G is a group scheme if there are maps*

1. $m : G \times_S G \to G$ *multiplication,*
2. $e : S \to G$ *unit section,*
3. $i : G \to G$ *inverse.*

satisfying the following axioms:

1. *associativity:*

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{(id,m)} & G \times G \\ \downarrow{\scriptstyle (m,id)} & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G \end{array}$$

2. *identity axiom:*

$$\begin{array}{ccc} G \times_S S \cong G & \xrightarrow{id} & G \\ & {\scriptstyle (id,e)} \searrow \quad \nearrow {\scriptstyle m} & \\ & G \times G & \end{array}$$

   and similarly for $S \times_S G \cong G$,

3. *inverse:*

$$\begin{array}{ccc} G & \xrightarrow{(id,i)} & G \times G \\ \downarrow & & \downarrow{\scriptstyle m} \\ S & \xrightarrow{e} & G \end{array}$$

Lemma 5.2. *Let G be an S-scheme. It is a group scheme if and only if $G(T)$ is a group functorial in T for all T/S.*

Proof: Yoneda's lemma.

Definition 5.3. *Let G, H be group schemes over S. A map $f : G \to H$ of S-schemes is a homomorphism if $G(T) \to H(T)$ is a group homomorphism for all T/S.*

We define $\ker(f)$ to be an S-group scheme such that

$$\ker(f)(T) = \ker(G(T) \to H(T)).$$

Equivalently, $\ker(f)$ is the fiber of the unit section.

Example 5.4. *The multiplication by n map $[n]_G : G \to G$ is defined by $g \mapsto g^n$.*

Assume $S = \mathrm{Spec}(R)$.

Definition 5.5. *Then $G = \mathrm{Spec}(A)$ is an R-group scheme if it has*

1. $\mu : A \to A \otimes_R A$ *comultiplication,*
2. $\varepsilon : A \to R$ *counit,*
3. $\iota : A \to A$ *coinverse*

that correspond to multiplication, unit section, and inverse.

Next time we will continue on finite flat group schemes.