p-adic Hodge Theory (Spring 2023): Week 1

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Introduction: A first glimpse of p-adic Hodge Theory

1 The arithmetic perspective

We start with an arithmetic perspective. The goal is to study p-adic representations, i.e. continuous representations

$$\Gamma_K = \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_p),$$

where K is a p-adic field. This is quite different from studying ℓ-adic representations, i.e. continuous representations

$$\Gamma_K = \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell), \quad \text{para } \ell \neq p.$$

Indeed, the topologies in this case are not quite compatible, so there are not as many representations as in the $\ell = p$ case.

To get started, let us consider a motivating example: Let E be an elliptic curve over $\mathbb{Q}_p$ with good reduction. There is an elliptic scheme $\mathcal{E}$ over $\mathbb{Z}_p$ such that $\mathcal{E}_{\mathbb{Q}_p} = E$. For a prime ℓ (which may or may not be equal to p), we define the *Tate module*

$$T_\ell(E) = \varprojlim E[\ell^n](\overline{\mathbb{Q}_p}) \cong \mathbb{Z}_\ell^2,$$

which has a continuous $\Gamma_{\mathbb{Q}_p}$-action. Tensoring with $\mathbb{Q}_\ell$, we get a continuous $\Gamma_{\mathbb{Q}_p}$-representation

$$V_\ell(E) = T_\ell(E) \otimes \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^2.$$

These representations see a lot of information about the elliptic curves. For example, we have the following fact.

Fact 1.1. *Given two elliptic curves E_1, E_2 over $\mathbb{Q}_p$, the natural maps*

$$\mathrm{Hom}(E_1, E_2) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}_{\Gamma_{\mathbb{Q}_p}}(T_\ell(E_1), T_\ell(E_2))$$

$$\mathrm{Hom}(E_1, E_2) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{Hom}_{\Gamma_{\mathbb{Q}_p}}(V_\ell(E_1), V_\ell(E_2))$$

are injective.

How to study $T_\ell(E)$? For $\ell \neq p$, we can consider the special fiber $\mathcal{E}_{\mathbb{F}_p}$, en elliptic curve over $\mathbb{F}_p$. The Tate module $T_\ell(\mathcal{E}_{\mathbb{F}_p})$ is a continuous $\Gamma_{\mathbb{F}_p}$-representation. To describe the action, it is enough to describe the action of Frobenius (a topological generator for $\Gamma_{\mathbb{F}_p}$): it acts on $T_\ell(\mathcal{E}_{\mathbb{F}_p})$ with characteristic polynomial $x^2 - ax + p$ where $a = p + 1 - \#(\mathcal{E}_{\mathbb{F}_p}(\mathbb{F}_p))$.

The punchline is that the reduction map

$$T_\ell(E) \to T_\ell(\mathcal{E}_{\mathbb{F}_p}) \tag{1}$$

is an isomorphism of $\Gamma_{\mathbb{Q}_p}$-representations, where the right hand side is a $\Gamma_{\mathbb{Q}_p}$ representation via the surjection $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p} \cong \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$. Therefore:

1. The action of Γ_{Q_p} factors throught the map $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$.
2. Frobenius of $\Gamma_{\mathbb{F}_p}$ acts with acharacteristic polynomial $x^2 - ax + p$.

The condition (1) is equivalent to the representation of $\Gamma_{\mathbb{Q}_p}$ being *unramified.*

Theorem 1.2 (Néron-Ogg-Shafarevich)**.** *An elliptic curve $E/\mathbb{Q}_p$ has good reduction if and only if $T_\ell(E)$ is unramified for all $\ell \neq p$.*

So what about $\ell = p$? Now we see that the key isomorphism (1) never holds. In fact,

$$T_p(\mathcal{E}_{\mathbb{F}_p}) \cong 0 \text{ or } \mathbb{Z}_p,$$

so it has the wrong rank. Let

$$I_{\mathbb{Q}_p} = \ker(\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}),$$

to be the *inertia* group. Then there is a non-trivial constribution from $I_{\mathbb{Q}_p}$. So how?
The solution to this problem was found by Grothendieck and Tate. We define

$$E[p^\infty] = \varinjlim E[p^n],$$

the p-*divisible group* of E. Note here it is a limit of schemes, not of the point of the schemes.

Fact 1.3. *We can recover the action of* $\Gamma_{\mathbb{Q}_p}$ *on* $T_p(E)$ *from* $E[p^\infty]$.

The schemes $\mathcal{E}[p^\infty]$ and $\mathcal{E}_{\mathbb{F}_p}[p^\infty]$ are also defined. In fact, we have the maps:

$$\begin{tikzcd} & \mathcal{E}[p^\infty] \arrow[dl, "\otimes\mathbb{Q}_p"'] \arrow[dr, "\otimes\mathbb{F}_p"] & \\ \mathcal{E}[p^\infty] & & \mathcal{E}_{\mathbb{F}_p}[p^\infty] \end{tikzcd}$$

Theorem 1.4 (Tate)**.** *The functor*

$$\{p\text{-divisible groups over } \mathbb{Z}_p\} \xrightarrow{\otimes\mathbb{Q}_p} \{p\text{-divisible groups over } \mathbb{Q}_p\}$$

is fully faithful.

Understanding the proof of the theorem and related results will be the goal of Week 3-6.

Theorem 1.5 (Dieudonné, Fontaine)**.** *There are equivalences of categories*

$$\{p\text{-divisible groups over } \mathbb{F}_p\} \longleftrightarrow \{\text{Dieudonné modules over } \mathbb{F}_p\},$$

$$\{p\text{-divisible groups over } \mathbb{Z}_p\} \longleftrightarrow \{p\text{-divisible groups over } \mathbb{F}_p \text{ with an ”admissible filtration”}\},$$

Definition 1.6. *A Dieudonné module over* $\mathbb{F}_p$ *means a finite free* $\mathbb{Z}_p$*-module* M *equipped with a (Frobeniussemilinear) endomorphism* φ *such that* $pM \subset \varphi(M)$.

One should think of $\mathbb{Z}_p$ here are the ring of Witt vectors of $\mathbb{F}_p$, $\mathbb{Z}_p = W(\mathbb{F}_p)$.
The following summarizes the situation:

$$T_p(E) \to E[p^\infty] \text{ } p\text{-divisible groups } \to \{\text{Dieudonné modules over } \mathbb{F}_p + \text{ extra data}\}.$$

After inverting p, we also get

$$V_p(E) \to \{\text{”isocrystals” over } \mathbb{F}_p + \text{ extra data }\}.$$

The general themes of p-adic Hodge theory are:

1. To construct a dictionary between certain p-adic representations and certain semilinear algebraic objects.
2. Change base field to $\widehat{\mathbb{Q}_p^{un}}$.

Since $\mathbb{Q}_p^{un}$ is not p-adically complete any more, we need to work with $\widehat{\mathbb{Q}_p^{un}}$ instead.
Many interesting properties of p-adic representatios are encoded in the action of $I_{\mathbb{Q}_p}$. We note that:

$$I_{\mathbb{Q}_p} = I_{\mathbb{Q}_p^{un}} = I_{\widehat{\mathbb{Q}_p^{un}}}.$$

Usually, base changing to $\widehat{\mathbb{Q}_p^{un}}$ simplifies things.
In the above correspondence, base changing to $\widehat{\mathbb{Q}_p^{un}}$ roughly corresponds to replacing $\mathbb{F}_p$ by $\overline{\mathbb{F}}_p$.

Theorem 1.7 (Manin)**.** *The category of isocrystals over* $\overline{\mathbb{F}}_p$ *are semisimple.*

Now **question** is: Is there a *general framework* or *formalism* that provides all these general themes in more general scope?
To properly answer this question, we need to discuss the geometric side of the story.

2 The geometric perspective

The goal here is to use p-adic representations to study the geometry of algebraic varieties X over K. We look at the cohomology of X:

- $H_{\text{ét}}$: étale cohomology
- H_{dR}: algebraic de Rham cohomology
- H_{cris}: crystalline cohomology

By definition, $H_{\text{ét}}$ is a p-adic Galois representation. The main goal is to find comparison theorems between the three cohomology theories.
In classical Hodge theory, there are many comparison theorems:

- between singular cohomology[1] and Hodge cohomology,
- between singular cohomology and de Rham cohomology

valid for proper smooth varieties over $\mathbb{C}$.
The reason for the name p-adic Hodge theory comes from the above motivation. The main issue in finding these comparison theorems is finding the correct *period ring*.
The obvious answer would be to work with $\widehat{\overline{K}}$, but we will soon see that this ring is not sufficient.
We first recall in more detail one of the comparison theorems from Hodge theory.

Theorem 2.1 (Hodge decomposition)**.** *Let Y be a proper smooth variety over $\mathbb{C}$. Then*

$$H^n(Y(\mathbb{C}), \mathbb{C}) \cong \bigoplus_{i+j=n} H^i(Y, \Omega_Y^j).$$

Corollary 2.1.1. *The Hodge number of Y are topological invariants.*

Let $C_K = \widehat{\overline{K}}$. IT has a continuous Γ_K-action. The *p-adic cyclotomic character* is

$$\chi : \Gamma_K \to \mathbb{Z}_p^\times$$

such that for any p-power root of unity ζ,

$$\sigma(\zeta) = \zeta^{\chi(\sigma)}$$

Definition 2.2. *We define a Tate twist as a Γ_K-representation $\mathbb{C}_K(j)$ with the underlying vector space $\mathbb{C}_K$ and $\sigma \in \Gamma_K$ acting by $\chi^k(\sigma) \cdot \sigma$.*

Theorem 2.3 (Hodge-Tate decomposition, Faltings)**.** *Let X be a proper smooth variety over K. Then*

$$H_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbb{C}_K(-j),$$

compatible with Γ_K-action, where

- *σ acts by $\sigma \otimes \sigma$ on the left hand side*
- *σ acts by $1 \otimes \sigma$ on the right hand side.*

Tate proved when X is an abelian variety with good reduction as a by product of the generic fiber functor theorem
Define the *Hodge–Tate period ring*

$$B_{HT} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}_K(j).$$

Then the Hodge-Tate decomposition in theorem 2.3 could be restated as

$$H_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{HT} \cong \left(\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \right) \otimes B_{HT},$$

Theorem 2.4 (Tate-Sen)**.** *We have that $B_{HT}^{\Gamma_K} = K$.*

[1] One should think that singular cohomology over $\mathbb{C}$ corresponds to étale cohomology in the p-adic setting

As a consequence, we see that

$$\left(H_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{HT}\right)^{\Gamma_K} = \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K})$$

Here is another result from Hodge theory. There is an isomorphism

$$H^n(Y(\mathbb{C}), \mathbb{C}) \cong H^n_{dR}(Y/\mathbb{C})$$

coming from the period pairing

$$H^n_{dR}(Y/\mathbb{C}) \times H_{2d-n}(Y, \mathbb{C}) \to \mathbb{C},$$

$$(\omega, \Gamma) \mapsto \int_\Gamma \omega.$$

Now the Goal is to construct a p-adic period ring.
Fontaine constructed a p-adic period ring B_{dR} such that:

1. B_{dR} carries Γ_K-action with $B_{dR}^{\Gamma_K} = K$,
2. B_{dR} carries a filtration with the associated graded ring B_{HT}.

Theorem 2.5 (Faltings)**.** *We have that*

$$H_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} \cong H^n_{dR}(X/K) \otimes_K B_{dR}$$

compatible with Γ_K actions and filtrations.

By construction, $H^n_{dR}(X/K)$ has a *Hodge filtration* such that the associated graded is

$$\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K})$$

The filtration on the right hand side of Faltings' Theorem 2.5 is given by the convolution filtration

$$Fil^m = \bigoplus_{a+b=m} Fil^a \otimes Fil^b$$

Remark 1. *We note that*

- *By passing to the associated graded in Faltings' Theorem 2.5, we recover the Hodge–Tate decomposition 2.3.*
- *We have that $(H_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_K} \cong H^n_{dR}(X/K)$.*
- *We will not attempt to prove Faltings' Theorem 2.5, but we will use it as motivation.*

Question: Is there a refinement of H_{dR} which recovers $H_{\text{ét}}$ itself?
Answer: Yes, cristalline cohomology H_{cris}.

Conjecture 2.6 (Grothendieck)**.** *Let $\mathcal{O}_K$ be the valuation ring of K and k be the residue field of $\mathcal{O}_K$. Let $W(k)$ be the ring of Witt vectors of k and $K_0 = Frac(W(k))$. (If $K = \mathbb{Q}_p$ then $K_0 = \mathbb{Q}_p$, and if K is a finite extension of $\mathbb{Q}_p$, then K_0 is the maximal unramified subextension.*
There should be a (purely algebraic) fully faithful functor D on a certain category of representations such that

$$D(H_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)) = H^n_{cris}(\mathcal{X}_K/W(k)) \otimes_{W(k)} K_0$$

for any proper smooth X with integral model X over $\mathcal{O}_K$.

Recall that for any elliptic curve E over $\mathbb{Q}_p$ with good reduction, we have seen that there is a fully faithful functor

$$V_p(E) \rightsquigarrow \{\text{filtered isocrystal}\}.$$

Now,

$$V_p(E) \cong \left(H^1_{\text{ét}}(E_{\overline{\mathbb{Q}_p}, \mathbb{Q}_p})\right)$$

and

$$\{\text{filtered isocrystal}\} \cong H^1_{cris}(\mathcal{E}_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Z}_p\mathbb{Q}_p.$$

Grothendieck's conjecture 2.6 is a generalization of this. By purely algebraic we mean that there should be a way to avoid going through p-divisible groups (which are geometric).
Fontaine constructed another period ring, called B_{cris} such that

1. B_{cris} carries an action of Γ_K such that $B^{\Gamma_K} = K_0$,
2. B_{cris} carries a semi-linear endomorphism φ called the *Frobenius action*,
3. there is a natural map $B_{cris} \otimes_{K_0} K \hookrightarrow B_{dR}$, inducing a filtration on B_{cris}.

Theorem 2.7 (Faltings)**.** *Suppose X has good reduction with integral model $\mathcal{X}$. Then*

$$H_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} \cong H^n_{cris}(\mathcal{X}_k/W(k)) \otimes B_{cris}$$

compatible with Γ_K-action, filtration, and Frobenius action.

Remark 2. *By construction, $H_{\acute{e}t}(\mathcal{X}_k/W(k))$ carries a Frobenius action. Frobenius acts only through B_{cris} on the left hand side and diagonally on the right hand side.*
The isomorphism

$$H_{\acute{e}t}(\mathcal{X}_k/W(k)) \otimes_{W(k)} K \cong H_{dR}(X/K)$$

gives a filtration on H_{cris}. We use the convolution filtration on the right hand side.

Now, taking Γ_K-invariants of both sides gives:

$$\left(H_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris}\right)^{\Gamma_K} \cong H^n_{cris}(\mathcal{X}_k/W(k)) \otimes_{W(k)} K_0$$

There is an *inverse functor* so we get D, *Grothendieck's mysterios functor*, given by

$$D(V) = (V \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K}.$$

This would prove Grothendieck's conjecture 2.6 if we define the domain of this functor and prove that it is fully faithful.

3 Interplay via representation theory

Fontaine built the formalism for functors that connect the geometric and arithmetic sides. This will be the focus of week 8 10.
Let B be any period ring such as B_{HT}, B_{dR}, B_{cris}. Then define

$$\text{Rep}_{\mathbb{Q}_p}(\Gamma_k) = \text{category of } p\text{-adic representations of } \Gamma_K.$$

Define $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$. A representation $V \in \text{Rep}_{\mathbb{Q}_p}(\Gamma_k)$ is B-admissible if the natural maps

$$(V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \otimes B \to V \otimes B$$

is an isomorphism.
Now, D_B defines a functor on $\text{Rep}_{\mathbb{Q}_p}(\Gamma_k)$, the category of B-admissible representations. The target category reflects the structure on B.

Example 3.1.
1. *If $B = B_{HT}$, the target category is the category of finite-dimensional graded vector space.*
2. *If $B = B_{dR}$, the target category is the category of finite-dimensional filtered vector space.*
3. *If $B = B_{cris}$, the target category is the category of finite-dimensional filtered vector spaces with Frobenius action*

Theorem 3.2 (Fontaine)**.** *The functors $D_{B_{HT}}$, $D_{B_{dR}}$, $D_{B_{cris}}$ are exact and faithful. Moreover, $D_{B_{cris}}$ is fully faithful.*

In particular, this proves Grothendieck's conjecture 2.6.

Next week, we will provide a gentle introduction to Fargues-Fontaine curve.