p-adic Hodge Theory (Spring 2023): Week 12

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This week: crystalline Representations (continued), introduction to Fargues-Fontaine curve

A quick update from last time: This is the theorem of falitings:

Theorem 0.1 (Faltings, 1988). *Suppose that X has good reduction, meaning that it has a proper smooth model $\mathcal{X}$ over $\mathcal{O}_K$. There exists a canonical isomorphism*

$$H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} \cong H^n_{cris}(\mathcal{X}_K/W(k))[1/p] \otimes_{K_0} B_{cris}$$

1 Properties of Crystalline Representations

Definition 1.1. *We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is crystalline if it is B_{cris}-admissible. We write $\mathrm{Rep}^{cris}_{\mathbb{Q}_p}(\Gamma_K) := \mathrm{Rep}^{B_{cris}}_{\mathbb{Q}_p}(\Gamma_K)$ for the category of crystalline p-adic Γ_K-representations. In addition, we write D_{cris} the functors $D_{B_{cris}}$.*

Example 1.2.

1. *Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$, since $\dim_K D_{cris}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$ and $D_{cris}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K}$ is nontrivial: contains $1 \otimes t^{-n}$.*
2. *For every proper smooth variety X over K with with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$, the étale cohomology $H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline by the theorem 0.1 above. moreover, there exists a canonical isomorphism*

   $$D_{cris}(H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H_{cris}(\mathcal{X}_K/K_0) = H^n_{cris}(\mathcal{X}_k/W(k))[1/p],$$

   where $H^n_{cris}(\mathcal{X}_K/W(k))$ denote the crystalline cohomology of $\mathcal{X}_k$.
3. *For every p-divisible group G over $\mathcal{O}_K$, the rational Tate module $V_p(G)$ is crystalline as proved by Fontaine; indeed, there exists a natural identification*

   $$D_{cris}(V_p(G)) \cong \mathbb{D}(\overline{G})[1/p]$$

   where $\mathbb{D}(\overline{G})$ denotes the Dieudonné module associated to $\overline{G} := G \times_{\mathcal{O}_K} k$.

Aim to promote D_{cris} to a functor that incorporates both the Frobenius endomorphism and the filtration on B_{cris}. Let us denote by σ the Frobenius automorphism of K_0.

Definition 1.3. *A filtered isocrystal (**filtered φ-modules**) over K is an isocrystal N over K_0 together with a collection of K-spaces $\{Fil^n(N_K)\}$ which yields a structure of a filtered vector space over K on $N_K := N \otimes_{K_0} K$. We denote by MF^{φ}_K the category of filtered isocrystals over K with the natural notions of morphisms, tensor products, and duals inherited from the corresponding notions for Fil_K and the category of isocrystals over K_0.*

Example 1.4. *The crystalline cohomology $H_{cris}(X_k/K_0) = H^n_{cris}(X_k/W(k))[1/p]$: a filtered isocrystal over K with the Frobenius automorphism $\varphi^*_{\overline{X}_k}$ induced by the relative Frobenius of $\mathcal{X}_K$ and the filtration on $H^n_{cris}(\mathcal{X}_k/K_0) \otimes_{K_0} K$ given by the Hodge filtration on the de Rham cohomology $H^n_{dR}(X/K)$ via the canonical comparison isomorphism*

$$H^n_{cris}(\mathcal{X}_k/K_0) \otimes_{K_0} K \cong H^n_{dR}(X/K).$$

Lemma 1.5. *The automorphism σ on K_0 extends to the endomorphism φ on B_{cris}.*

Lemma 1.6. *Let N be a finite dimensional vector space over K_0. Every injective σ-semilinear additive map $f : N \to N$ is bijective.*

Proposition 1.7. *Let V be a p-adic representation of Γ_K. Then $D_{cris}(V) = (V \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K}$ is naturally a filtered isocrystal over K with the Frobenius automorphism $1 \otimes \varphi$ and the filtration on $D_{cris}(V)_K = D_{cris}(V) \otimes_{K_0} K$ given by*

$$Fil^n(D_{cris}(V)_K) := (V \otimes_{\mathbb{Q}_p} Fil^n(B_{cris} \otimes_{K_0} K))^{\Gamma_K}.$$

Proof. Since Γ_K acts trivially on K, we have a natural identification

$$D_{cris}(V)_K = (V \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K} \otimes_{K_0} K = (V \otimes_{\mathbb{Q}_p} (B_{cris} \otimes_{K_0} K))^{\Gamma_K}.$$

$D_{cris}(V)_K$ is a filtered vector space over K with the filtration $Fil^n(D_{cris}(V)_K)$ as defined above. Therefore it remains to verify that the map $1 \otimes \varphi$ is σ-semilinear and bijective on $D_{cris}(V)$. For arbitrary $v \in V$, $b \in B_{cris}$, and $c \in K_0$ we have

$$(1 \otimes \varphi)(c(v \otimes b)) = (1 \otimes \varphi)(v \otimes bc) = v \otimes \varphi(b)\varphi(c) = \varphi(c) \cdot (1 \otimes \varphi)(v \otimes b).$$

Hence by Lemma 1.5 we find that the additive map $1 \otimes \varphi$ is σ-semilinear. Moreover, the map $1 \otimes \varphi$ is injective on $D_{cris}(K)$ and the left exactness of the functor D_{cris}. Thus we deduce the desired assertion by Lemma 1.6. □

Proposition 1.8. *Let V be a crystalline representation of Γ_K. Then V is de Rham with a natural isomorphism of filtered vector spaces*

$$D_{cris}(V)_K = D_{cris}(V) \otimes_{K_0} K \cong D_{dR}(V).$$

Proof. Proposition 1.5 and Proposition 1.6 from last week together imply that the natural map $B_{cris} \otimes_{K_0} K \to B_{dR}$ identifies $B_{cris} \otimes_{K_0} K$ as a filtered subspace of B_{dR} over K; in other words, we have an identification

$$Fil^n(B_{cris} \otimes_{K_0} K) = (B_{cris} \otimes_{K_0} K) \cap Fil^n(B_{dR}) \text{ for every } n \in \mathbb{Z}.$$

Therefore Proposition 1.7 yields a natural injective morphism of filtered vector spaces

$$D_{cris}(V)_K = (V \otimes_{\mathbb{Q}_p} (B_{cris} \otimes_{K_0} K))^{\Gamma_K} \hookrightarrow (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_K} = D_{dR}(V)$$

with an identification

$$Fil^n(D_{cris}(V) \otimes_{K_0} K) = (D_{cris}(V) \otimes_{K_0} K) \cap Fil^n(D_{dR}(V)) \text{ for every } n \in \mathbb{Z}$$

We then find

$$\dim_{K_0} D_{cris}(V) = \dim_K D_{cris}(V)_K \leq \dim_K D_{dR}(V) \leq \dim_{\mathbb{Q}_p} V$$

where the last inequality follows from Theorem 1.2.1. Since V is crystalline, both inequalities should be in fact equalities, thereby yielding the desired assertion. □

Example 1.9. $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ *be a nontrivial continuous character which factors through $Gal(L/K)$ for some totally ramified finite extension L of K. Then $\mathbb{Q}_p(\eta)$ is de Rham. We assert that $\mathbb{Q}_p(\eta)$ is not crystalline. Let us write Γ_L for the absolute Galois group of L. Since L is totally ramified over K, we have $B_{cris}^{\Gamma_L} \cong K_0$ by Theorem 1.8 from last week and the fact that the construction of B_{cris} depends only on $\mathbb{C}_K$. Moreover, we have $\mathbb{Q}_p(\eta)^{\Gamma_L} = \mathbb{Q}_p(\eta)$ and $\mathbb{Q}_p(\eta)^{Gal(L/K)} = 0$ by construction. Hence we find an identification*

$$\begin{aligned} D_{\text{cris}}(\mathbb{Q}_p(\eta)) &= (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K} = \left((\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_L}\right)^{\text{Gal}(L/K)} \\ &= \left(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\text{cris}}^{\Gamma_L}\right)^{\text{Gal}(L/K)} \cong (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} K_0)^{\text{Gal}(L/K)} \\ &= \mathbb{Q}_p(\eta)^{\text{Gal}(L/K)} \otimes_{\mathbb{Q}_p} K_0 = 0, \end{aligned}$$

thereby deducing the desired assertion.

Below prop: shows general formalism extends to the category of crystalline representations with enhanced functor D_{cris} that takes values in MF_K^φ.

Proposition 1.10. *Every $V \in \text{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$ induces a natural Γ_K-equivariant isomorphism*

$$D_{cris}(V) \otimes_{K_0} B_{cris} \cong V \otimes_{\mathbb{Q}_p} B_{cris}$$

which is compatible with the natural Frobenius endomorphisms on both sides and induces a K-linear isomorphism of filtered vector spaces

$$D_{cris}(V)_K \otimes_K (B_{cris} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{cris} \otimes_{K_0} K)$$

Proof. Since V is crystalline, the natural map

$$D_{cris} \otimes_{K_0} B_{cris} \to (V \otimes_{\mathbb{Q}_p} B_{cris}) \otimes_{K_0} B_{cris} \cong V \otimes_{\mathbb{Q}_p} (B_{cris} \otimes_{K_0} K) \to V \otimes_{\mathbb{Q}_p} B_{cris}$$

is a Γ_K-equivariant B_{cris}-linear isomorphism. Moreover, this map is visibly compatible with the natural Frobenius endomorphisms on $D_{cris}(V) \otimes_{K_0} B_{cris} = (V \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K} \otimes_{K_0} B_{cris}$ and $V \otimes_{\mathbb{Q}_p} B_{cris}$ respectively given by $1 \otimes \varphi \otimes \varphi$ and $1 \otimes \varphi$. Let us now consider the induced K-linear bijective map

$$(D_{cris}(V))_K \otimes_K (B_{cris} \otimes_{K_0} K) \to V \otimes_{\mathbb{Q}_p} (B_{cris} \otimes_{K_0} K).$$

It is straightforward to check that this map is a morphism of filtered vector spaces. Therefore by Proposition 1.8 from last week it suffices to show that the induced map

$$gr(D_{cris}(V)_K \otimes_K (B_{cris} \otimes_{K_0} K)) \to gr(V \otimes_{\mathbb{Q}_p} (B_{cris} \otimes_{K_0} K)) \tag{1}$$

is an isomorphism. As V is crystalline, it is also Hodge-Tate with the natural isomorphism of graded vector space

$$gr(D_{cris}(V)_K) \cong gr(D_{dr}(V)) \cong D_{HT}(V)$$

by Proposition 1.8 last week and Proposition 2.4.4 from weeks ago. Hence Proposition 2.3.10 and Proposition 1.7 from two weeks ago together yield identification

$$\begin{aligned} \mathrm{gr}\,(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)) &\cong \mathrm{gr}(D_{\mathrm{cris}}(V)_K) \otimes_K \mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}}, \\ \mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K)\right) &\cong V \otimes_{\mathbb{Q}_p} \mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}. \end{aligned}$$

We thus identify the map (1) with the natural map

$$D_{HT}(V) \otimes_K B_{HT} \to V \otimes_{\mathbb{Q}_p} B_{HT}$$

given by Theorem 1.2.1, thereby deducing the desired assertion by the fact that V is Hodge-Tate. □

Similar to the proof last time, we have the following proposition:

Proposition 1.11. *The functor D_{cris} with values in MF_K^{φ} is faithful and exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$.*

And an immediate cosequence of the above proposition, as well as the fact that $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_K)$ is closed under taking subquotients, we have that

Corollary 1.12. *Let V be a crystalline representation. Every subquotient W of V is a crystalline representation with $D_{cris}(W)$ naturally identified as a subquotient of $D_{dR}(V)$.*

Proposition 1.13. *Given any $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$, we have $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$ with a natural isomorphism of filtered isocrystals*

$$D_{cris}(V) \otimes_{K_0} D_{cris}(W) \cong D_{cris}(V \otimes_{\mathbb{Q}_p} W).$$

Proof. $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$ isomorphism of vector space from B-admissibility. The multiplicative structure of B_{cris} shows that the map is a morphism of isocrystals over K_0. Now proposition 1.8 implies that we can identify the induced bijective K-linear map

$$D_{cris}(V)_K \otimes_K D_{cris}(W)_K \to D_{cris}(V \otimes_{\mathbb{Q}_p} W)_K.$$

with the natural isomorphism of filtered vector spaces

$$D_{dR}(V) \otimes_K D_{dR}(W)_K \cong D_{dR}(V \otimes_{\mathbb{Q}_p} W)$$

Therefore we deduce that the map (1) is an isomorphism in MF_K^{φ} as desired. □

Proposition 1.14. *For every crystalline representation V, we have $\wedge^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$ and $\mathrm{Sym}^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$ with natural isomorphisms of filtered isocrystals*

$$\wedge^n(D_{cris}(V)) \cong D_{cris}(\wedge^n(V)) \quad \text{and} \quad \mathrm{Sym}^n(D_{cris}(V)) \cong D_{cris}(\mathrm{Sym}^n(V))$$

Proposition 1.15. *For every crystalline representation V, the dual representation $V^\vee$ is crystalline with a natural perfect pairing of filtered isocrystals*

$$D_{cris}(V) \otimes_{K_0} D_{cris}(V^\vee) \cong D_{cris}(V \otimes_{\mathbb{Q}_p} V^\vee) \to D_{cris}(\mathbb{Q}_p).$$

Definition 1.16. *Let M be a module over a ring R with an additive endomorphism f. For every $r \in R$, we refer to the subgroup*

$$M^{f=r} := \{m \in M : f(m) = rm\}$$

as the eigenspace of f with eigenvalue r.

Lemma 1.17. *We have an identification*

$$B_{cris}^{\varphi=1} \cap Fil^0(B_{cris} \otimes_{K_0} K) = B_{cris}^{\varphi=1} \cap B_{dR}^+ = \mathbb{Q}_p.$$

Proof. By Proposition 1.6 and Theorem 1.14 from last week we find

$$B_{cris}^{\varphi=1} \cap Fil^0(B_{cris} \otimes_{K_0} K) \subseteq B_{cris}^{\varphi=1} \cap Fil^0(B_{dR}) = B_{cris}^{\varphi=1} \cap B_{dR}^+ = \mathbb{Q}_p,$$

and thus obtain the desired identification as both $B_{cris}^{\varphi=1}$ and $Fil^0(B_{cris} \otimes_{K_0} K)$ contain $\mathbb{Q}_p$. □

Proposition 1.18. *Every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$ admits canonical isomorphisms*

$$\begin{aligned} V &\cong (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cap \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \\ &\cong (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cap \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K B_{\mathrm{dR}}\right). \end{aligned}$$

Proof. Proposition 1.10 yields a natural Γ_K-equivariant isomorphism

$$D_{cris}(V) \otimes_{K_0} B_{cris} \cong V \otimes_{\mathbb{Q}_p} B_{cris}$$

which is compatible with the natural Frobenius endomorphisms on both sides and induces an isomorphism of filtered vector spaces

$$D_{cris}(V)_K \otimes_K (B_{cris} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{cris} \otimes_{K_0} K).$$

In addition, there exists a canonical isomorphism of filtered vector spaces

$$D_{cris}(V)_K \otimes_K B_{dR} \cong D_{dR}(V) \otimes_K B_{dR} \cong V \otimes_{\mathbb{Q}_p} B_{dR}$$

given by Proposition 1.8 and Proposition 2.4.8. Therefore we have identifications

$$\begin{aligned} (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} &\cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}^{\varphi=1}, \\ \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) &\cong V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K), \\ \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K B_{\mathrm{dR}}\right) &\cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+. \end{aligned}$$

The desired assertion now follows by Lemma 1.17. □

Theorem 1.19 (Fontaine)**.** *The functor D_{cris} with values in MF_K^{φ} is exact and fully faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$.*

Proof. By Proposition 1.11 we only need to establish the fullness of D_{cris} on $\mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$. Let V and W be arbitrary crystalline representations. Consider an arbitrary morphism $f : D_{cris}(V) \to D_{cris}(W)$ in MF_{φ}^K. Then f gives rise to a Γ_K-equivariant map

$$V \otimes_{\mathbb{Q}_p} B_{cris} \cong D_{cris}(V) \otimes_{K_0} B_{cris} \xrightarrow{f \otimes 1} D_{cris}(W) \otimes_{K_0} B_{cris} \cong W \otimes_{\mathbb{Q}_p} B_{cris} \tag{2}$$

where the isomorphisms are given by Proposition 1.10. Moreover, Proposition 1.18 implies that this map restricts to a linear map $\phi : V \to W$. In other words, we may identify the map (2) as $\phi \otimes 1$. In particular, since the isomorphisms in (2) are Γ_K-equivariant, we recover f as the restriction of $\phi \otimes 1$ on $(V \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K} \cong (D_{cris}(V) \otimes_{K_0} B_{cris})^{\Gamma_K} \cong D_{cris}(V)$. This precisely means that f is induced by ϕ via the functor D_{cris}. □

Proposition 1.20. *Let V be a p-adic representation of Γ_K. Let L be a finite unramified extension of K with the residue field extension ℓ of k. Denote by Γ_L the absolute Galois group of L and by L_0 the fraction field of the ring of Witt vectors over ℓ.*

1. *There exists a natural isomorphism of filtered isocrystals*

$$D_{cris,K}(L)\otimes_{K_0} L_0 \cong D_{cris,L}(V)$$

where we set $D_{cris,K}(V) := (V\otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K}$ *and* $D_{cris,L} := (V\otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K}$.

2. V *is crystalline if and only if it is crystalline as a representation of* Γ_L.

Proof. We only need to prove the first statement, as the second statement immediately follows from the first statement. By definition L and L_0 are respectively unramified extensions of K and K_0 with the residue field extension ℓ of k. Hence L and L_0 are respectively Galois over K and K_0 with $Gal(L/K)\cong Gal(L_0/K_0)$. Furthermore, since the construction of B_{cris} depends only on $\mathbb{C}_K$, we have an identification

$$D_{cris,K}(V) = D_{cris,L}(V)^{Gal(L/K)} = D_{cris,L}(V)^{Gal(L_0/K_0)}$$

Then by the Galois descent for vector spaces we obtain a natural bijective L_0-linear map

$$D_{cris,K}(V)\otimes_{K_0} L_0 \to D_{cris,L}(V). \tag{3}$$

This map is evidently compatible with the natural Frobenius automorphisms on both sides induced by φ as explained in Lemma 1.5 and Proposition 1.7. Moreover, Proposition 2.4.14 and Proposition 1.8 together imply that the map (3) induces a natural L-linear isomorphism of filtered vector spaces

$$(D_{cris,K}(V)\otimes_{K_0} K)\otimes_K L \cong D_{cris,L}(V)\otimes_{L_0} L.$$

We thus deduce that the map (3) is an isomorphism of filtered isocrystals over L. □

Remark. Proposition 1.20 also holds when L is the completion of the maximal unramified extension of K. As a consequence, we have the following facts:

1. Every unramified p-adic representation is crystalline.
2. For a continuous character $\eta : \Gamma_K \to \mathbb{Z}_p^\times$, we have $\mathbb{Q}_p(\eta)\in \mathrm{Rep}_{\mathbb{Q}_p}^{cris}(\Gamma_K)$ if and only if there exists some $n\in\mathbb{Z}$ such that $\eta\chi^n$ is trivial on I_K.

On the other hand, Example 1.9 shows that Proposition 1.20 fails when L is a ramified extension of K. Fontaine interpreted this "failure" as a good feature of the crystalline condition, and conjectured that the crystalline condition should provide a p-adic analogue of the Néron-Ogg-Shafarevich criterion introduced in Theorem 1.1.1 of Chapter I; more precisely, Fontaine conjectured that an abelian variety A over K has good reduction if and only if the rational Tate module $V_p(A[p^\infty])$ is crystalline. Fontaine's conjecture is now known to be true by the work of Coleman-Iovita and Breuil.

Proposition 1.21. *The continuous map* $\log : \mathbb{Z}_p(1)\to B_{dR}^+$ *extends to a* Γ_K*-equivariant homomorphism* $\log : A_{inf}[1/p]^\times \to B_{dR}^+$ *such that* $\log([p^\flat])$ *is transcendental over the fraction field of* B_{cris}.

Example 1.22. *The Tate curve* E_p *is an elliptic curve over* K *with* $E_p(K)\cong K^\times/p^{\mathbb{Z}}$ *where we set* $p^{\mathbb{Z}} := \{p^n : n\in\mathbb{Z}\}$. *We assert that the rational Tate module* $V_p(Ep[p^\infty])$ *is de Rham but not crystalline. It is evident by construction that* ε *and* $p^\flat$ *form a basis of* $V_p(Ep[p^\infty])$ *over* $\mathbb{Q}_p$. *Moreover, for every* $\gamma\in\Gamma_K$ *we have*

$$\gamma(\varepsilon) = \varepsilon^{\chi(\gamma)} \quad \text{and} \quad \gamma(p^\flat) = p^\flat\varepsilon^{c(\gamma)} \tag{4}$$

for some $c(\gamma)\in\mathbb{Z}_p$. *Hence* $V_p(E_p[p^\infty])$ *is an extension of* $\mathbb{Q}_p$ *by* $\mathbb{Q}_p(1)$ *in* $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, *and thus is de Rham by Example 2.4.5.*

We aim to present a basis for $D_{dR}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty])\otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_K}$. *By (4) we find* $\epsilon\otimes t^{-1}\in D_{dR}(V_p(E_p[p^\infty]))$. *Let us now consider the homomorphism* $\log : A_{inf}[1/p]^\times\to B_{dR}^+$ *as in Proposition 1.21 and set* $u := \log([p^\flat])$. *Then for* $\gamma\in\Gamma_K$ *we find*

$$\gamma(u) = \gamma(\log[p^\flat]) = \log([\gamma p^\flat]) = \log([p^\flat\varepsilon^{c(\gamma)}]) = \log([p^\flat]) + c(\gamma)\log([\varepsilon]) = u + c(\gamma)t$$

by (4) and Lemma 2.2.20, and consequently obtain

$$\begin{aligned}\gamma(-\varepsilon\otimes ut^{-1} + p^\flat\otimes 1) &= -\varepsilon^{\chi(\gamma)}\otimes(u+c(\gamma)t)\chi(\gamma)^{-1}t^{-1} + p^\flat\varepsilon^{c(\gamma)}\otimes 1\\ &= -\varepsilon\otimes(ut^{-1}+c(\gamma)) + c(\gamma)\cdot(\varepsilon\otimes 1) + p^\flat\otimes 1\\ &= -\varepsilon\otimes ut^{-1} + p^\flat\otimes 1\end{aligned}$$

by (3.13) and Theorem 2.2.21. In particular, we have $-\varepsilon \otimes ut - 1 + p^\flat \otimes 1 \in DdR(Vp(Ep[p^\infty]))$*. Since the elements* $\varepsilon \otimes t^{-1}$ *and* $-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1$ *are linearly independent over* B_{dR}*, they form a basis for* $D_{dR}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty]) \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_K}$ *. Let us now consider an arbitrary element* $x \in D_{cris}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty]) \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_K}$ *. We may uniquely write* $x = \varepsilon \otimes c + p^\flat \otimes d$ *for some* $c, d \in B_{cris}$*. Moreover, since we have* $D_{cris}(V_p(E_p[p^\infty])) \subseteq D_{dR}(V_p(E_p[p^\infty]))$ *there exist some* $r, s \in K$ *with*

$$x = r \cdot (\varepsilon \otimes t^{-1}) + s \cdot (-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1) = \varepsilon \otimes (r - su)t^{-1} + p^\flat \otimes s.$$

Then we find $c = (r - su)t^{-1}$*, and consequently obtain* $s = 0$ *by Proposition 1.21. Therefore we deduce that every element in* $D_{cris}(V_p(E_p[p^\infty])) \otimes_{K_0} K$ *is a* K*-multiple of* $\varepsilon \otimes t^{-1}$*. In particular, we find* $\dim_{K_0} D_{cris}(V_p(E_p[p^\infty])) \leq 1$*, thereby concluding that* $V_p(E_p[p^\infty])$ *is not crystalline.*

Remark. Fontaine constructed the semistable period ring B_{st} as the B_{cris}-subalgebra of B_{dR} generated by $\log([p^\flat])$.